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ORIGINAL ARTICLE

# A heuristic technique for inventory replenishment policy with increasing demand pattern and shortage allowance

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**Abstract** In the growth stage of a product life cycle, the demand rate is usually unstable and follows an increasing pattern. The traditional inventory policies, which have been developed for stationary demand pattern, are not appropriate to this situation. Although there exist some researches in the past dealing with inventory policy for the case of increasing demand pattern, most of them focused on the inventory systems in which shortages are not allowed. In reality, the presence of shortages is sometimes economically preferable when holding cost is significant as compared with shortage cost. The aim of the research presented in this paper is, therefore, to develop a replenishment policy for inventory systems with nonlinear increasing demand pattern and shortage allowance in such a way that the total demand during a predefined planning horizon can be exactly met. A heuristic technique to help determine the operational parameters for the inventory policy is then developed. In the proposed heuristic technique, the consecutive improvement method developed by Wang (*Comput Oper Res*, 29:1819–1825, 2002) will first be used to help determine replenishment times. And then, a new concept of reduction cost, which is defined as the difference between the holding cost when shortage is allowed and the incurred shortage cost, is introduced and applied to help find the optimal shortage starting point in each replenishment cycle. Numerical experiments are also conducted to illustrate the applicability of the proposed technique.

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## 1 Introduction

The life cycle of any products usually consists of four stages, which are introduction, growth, maturity, and decline (Cox [1]; Golder and Tellis [2]). Each stage in the product life cycle has its own characteristic demand pattern; for example, the growth stage has increasing demand pattern, the maturity stage has stable demand pattern, and the decline stage has decreasing demand pattern. For different stages of the product life cycle, different inventory policies should be employed. Moreover, as the life cycle of the product is actually finite, the inventory policy should be developed in the frame of a finite planning horizon (Diponegoro and Sarker [3]).

Related to the inventory policy for the case of linear increasing demand pattern, the initial work was done by Resh et al. [4]. Donaldson [5] also developed another analytical method for the same situation that later on has inspired many researchers to develop other heuristic methods. In 1993, Hariga [6] proposed an algorithm based on iterative numerical procedures to obtain optimal solution. Recently, Lo et al. [7] also proposed an exact solution technique using a two-equation model, which provided exactly the same result as in Donaldson's approach.

The first heuristic approach for determining inventory replenishment policy in the case of linear increasing demand pattern was developed by Silver [8], which was a modification of the Silver–Meal heuristic. This approach was tested against Donaldson's technique, and it was shown that the resulting total inventory cost was just slightly higher than the optimal one. Another heuristic technique was developed by Phelps [9] with the use of a constant replenishment period. It has been shown from numerical experiments that the total cost that resulted from Phelps' approach was typically only a few percent higher than that of the optimal solution. Lately, Teng [10] proposed a hybrid heuristic technique to help determine the inventory policy for linear increasing demand pattern. Mitra et al. [11] and Ritchie [12] also adjusted the economic order quantity model and applied it in the case of linear trend in demand pattern. In general, the heuristic methods discussed above can provide the total cost that is only slightly higher than the exact solution, except for the method of Teng [10] that usually provides the solution that is exactly the same as the optimal solution found by Donaldson's approach.

Related to the case of nonlinear increasing demand pattern, Yang et al. [13] proposed an analytic eclectic method for replenishment decision, and it had been illustrated that this method outperforms other heuristic methods, such as Silver's method, simple eclectic, continuous least unit cost, or continuous time part period methods. Another method for the case of nonlinear increasing demand pattern was developed by Wang [14]. He proposed a consecutive improvement method which is easy in concept, understandable, and computationally simple. The main idea of the consecutive improvement itself is to check if there are any possibilities that can help to reduce the total cost. By applying this method, the last replenishment interval will end exactly at the ending point of the pre-established planning horizon so that the demand over the pre-established time frame can always be met. This is the major advantage of Wang's method, as if the replenishment interval does not match exactly with the planning horizon, it results in so much penalty cost. Numerical examples have illustrated that Wang's consecutive improvement method provided better cost performance in comparison with Yang's eclectic and Silver's heuristic techniques. However, it should be noted that the heuristic technique proposed by Wang [14] deals only with the inventory systems where shortages are not allowed. In addition, it has no possibility to extend it to deal with the case of decreasing demand.

In reality, there are some situations where shortage might be preferable from the economic point of view, for example, when the inventory holding cost is significant compared to the cost of shortage. Zhou et al. [15] stated that the inventory models that deal with shortage basically can be divided into two categories, i.e., (1) IFS (inventory followed by shortages) in which every replenishment cycle starts with replenishment and ends with shortage (and except in the last cycle, no shortages are allowed to be happened), (2) SFI (shortage followed by inventory) in which every cycle starts with a shortage period before replenishment is received. Teng et al. [16] also discussed about the above two categories in their research. It should be noted that the first work, which is aimed at finding the optimal solution for linear increasing demand pattern considering shortage, was done by Murdeshwar [17], who used a similar approach as Donaldson [5]. Dave [18] also developed an approach in this direction. Another research was conducted by Hariga [19] who proposed an optimal solution procedure that can give a unique optimal replenishment schedule. This proposed method was then tested by using Dave's example [18], which is the extension of Donaldson's example when shortage is incorporated, and it showed a better cost performance. However, it should be noted that Hariga's method can not guarantee that the last replenishment interval will match with the ending point of the planning horizon. There still exists shortage that is not fulfilled in the last replenishment cycle.

Instead of aiming at the exact solution, many researchers have developed heuristic methods for determining the inventory policies with increasing demand pattern considering shortage. Deb and Chaudhuri [20] extended the Silver's heuristic to permit shortage that was completely backlogged. In addition, they also extended the model of Donaldson by considering shortage as well. In another research, Dave [21] pointed out that there exist some flaws in Deb and Chaudiri's method. Dave, then, has corrected and tested the adjusted method using Donaldson's examples. The result showed that the adjusted method provided the total cost, which was very close to the exact solution established by Dave [18].

For the case of inventory systems with shortage allowance and nonlinear increasing demand pattern, Yang et al. [22, 23] proposed forward and backward recursive algorithms for power-form demand pattern. However, the forward recursive algorithm was developed only for the inventory system that starts with shortage and also ends with shortage in the planning horizon. It should also be noted in the above two researches that, although shortages were assumed to be completely backlogged, the existence of shortage at the end of the planning horizon implies that the total demand could not be met. This might cause excess penalty cost in practical application. Recently, Chen et al. [24] also proposed a search procedure based on the Nelder–Mead algorithm to help find the solution for the case of inventory systems with shortage allowance and nonlinear demand pattern. In their research, the whole product life cycle can be taken into consideration by the assumption that the demand rate is a revised version of the Beta distribution function.

The focus of the research presented in this paper is to develop an algorithm for solving the inventory replenishment problem with shortage and nonlinear increasing demand pattern in such a way that the last replenishment interval will match with the pre-established planning horizon; i.e., no shortage will occur at the end of the planning horizon. The objective is to determine the timings of replenishment $\{t_i\}$ and the shortage starting points $\{s_i\}$ in a predefined planning horizon so as to minimize the total inventory cost. In this research, the timing of replenishment $\{t_i\}$ will be determined from the approach proposed by Wang [14]. And then, the concept of reduction cost will be introduced and applied to help determine the shortage starting points in all replenishment cycles, except for the last one. It is noted that, with the exclusion of the last replenishment cycle in the process of finding shortage starting points, i.e., no shortage is allowed in this cycle, the proposed technique can guarantee that the accumulated demand over the pre-established planning horizon will always be satisfied.

The remaining parts of this paper are arranged as follows. Section 2 presents the mathematical development in which the expressions for calculation of cumulative holding inventory, cumulative shortage, and the total cost function will be derived. In Section 3, the procedure proposed by Wang [14] to help find the replenishment times for the whole planning horizon will be first reviewed, and then, the proposed technique for determining the shortage starting points in each replenishment cycle based on reduction cost will be discussed. Numerical experiments are conducted in Section 4 to illustrate the applicability of the proposed method, and some concluding remarks will be discussed in Section 5.

## 2 Mathematical model

The following notation will be used throughout the paper:

- $H$ — length of the planning horizon under consideration
- $f(t)$ — instantaneous demand rate at time $t$
- $c_1$ — ordering cost per order
- $c_2$ — holding cost per unit per unit time
- $c_3$ — shortage cost per unit per unit time
- $n$ — number of replenishments
- $t_i$ — the $i$th replenishment time ($i = 1, 2, \ldots, n$)
- $s_i$ — the shortage starting point of cycle $i$, which is the time at which the inventory level reaches zero in the $i$th cycle $[t_i, t_{i+1}]$; ($i = 1, 2, \ldots, n-1$)
- $I(t)$ — inventory level at time $t$, which is evaluated after the replenishment arrives at time $t_i$ and before the replenishment arrives at time $t_{i+1}$ in the $i$th cycle $[t_i, t_{i+1}]$

The behavior of inventory level function of the inventory problem dealt with in this paper is presented in Fig. 1. For the development of the mathematical model, the following assumptions are used:

(a) The replenishment is made at time $t_i$ ($i = 1, 2, \ldots, n$) in which $t_1 = 0$
(b) Lead time is negligible; i.e., replenishment is instantaneous
(c) The quantity received at $t_i$ is used partly to meet accumulated shortages in the previous cycle from time $s_{i-1}$ to $t_i$ ($s_{i-1} < t_i$); ($i = 2, \ldots, n$)
(d) No shortages at the beginning ($t_1 = 0$) and the end of the planning horizon ($s_n = H$)

Using the assumptions above, the expression for the total relevant cost, which includes ordering cost, holding cost, and shortage cost, of the inventory system during the planning horizon $H$ when $n$ orders are placed is expressed as follows:

$$C(n, \{s_i\}, \{t_i\}) = nc_1 + c_2 \sum_{i=1}^{n} I_i + c_3 \sum_{i=1}^{n-1} S_i \qquad (1)$$

in which

- $I_i$ — cumulative holding inventory during cycle $i$
- $S_i$ — cumulative shortage during cycle $i$

For each cycle $i$ from $t_i$ to $t_{i+1}$, the cumulative holding inventory $I_i$ and cumulative shortage $S_i$ can be derived as in Sections 2.1 and 2.2 below.

### 2.1 Cumulative holding inventory $I_i$

If we denote $F(t)$ to be the cumulative demand up to time $t$ ($t_i \le t \le t_{i+1}$) then

$$F(t) = \int_0^t f(t)dt$$


**Fig. 1** Inventory level over the whole planning horizon

It is also noted that

$$f(t) = -\frac{dI(t)}{dt} \text{ and hence, } I(t) = \int_t^{s_i} f(\tau)d\tau \quad (t_i \le t \le s_i)$$

The cumulative inventory $I_i$ in cycle $i$ can then be determined as

$$\begin{aligned} I_i = \int_{t_i}^{s_i} I(t)dt &= \int_{t_i}^{s_i}\int_t^{s_i} f(\tau)d\tau dt \\ &= t\int_t^{s_i} f(\tau)d\tau\Big|_{t_i}^{s_i} + \int_{t_i}^{s_i} tf(t)dt \\ &= -t_i\int_{t_i}^{s_i} f(\tau)d\tau + \int_{t_i}^{s_i} tf(t)dt \\ &= -t_i[F(s_i) - F(t_i)] + \int_{t_i}^{s_i} tf(t)dt \end{aligned}$$

We also have

$$\begin{aligned} \int_{t_i}^{s_i} tf(t)dt = tF(t)\Big|_{t_i}^{s_i} - \int_{t_i}^{s_i} F(t)dt \\ = s_iF(s_i) - t_iF(t_i) - \int_{t_i}^{s_i} F(t)dt \end{aligned}$$

Hence,

$$I_i = (s_i - t_i)F(s_i) - \int_{t_i}^{s_i} F(t)dt \tag{2}$$

## 2.2 Cumulative shortage $S_i$

The cumulative shortage $S_i$ in cycle $i$ ($i$ = 1,2...,$n$–1) can be determined as

$$\begin{aligned} S_i = \int_{s_i}^{t_{i+1}} (-I(t))dt &= \int_{s_i}^{t_{i+1}}\int_{s_i}^{t} f(\tau)d\tau dt \\ &= t\int_{s_i}^{t} f(\tau)d\tau\Big|_{s_i}^{t_{i+1}} - \int_{s_i}^{t_{i+1}} tf(t)dt \\ &= t_{i+1}\int_{s_i}^{t_{i+1}} f(\tau)d\tau - \int_{s_i}^{t_{i+1}} tf(t)dt \\ &= t_{i+1}[F(t_{i+1}) - F(s_i)] - \int_{s_i}^{t_{i+1}} tf(t)dt \end{aligned}$$

We have

$$\begin{aligned} \int_{s_i}^{t_{i+1}} tf(t)dt = tF(t)\Big|_{s_i}^{t_{i+1}} - \int_{s_i}^{t_{i+1}} F(t)dt \\ = t_{i+1}F(t_{i+1}) - s_iF(s_i) - \int_{s_i}^{t_{i+1}} F(t)dt \end{aligned}$$

Hence,

$$S_i = (s_i - t_{i+1})F(s_i) + \int_{s_i}^{t_{i+1}} F(t)dt \tag{3}$$

From Eqs. 2 and 3, the expression of the total cost can be defined as follows:

$$\begin{aligned} C(n, \{s_i\}, \{t_i\}) = nc_1 + c_2\sum_{i=1}^{n}\left\{(s_i - t_i)F(s_i) - \int_{t_i}^{s_i} F(t)dt\right\} \\ + c_3\sum_{i=1}^{n-1}\left\{(s_i - t_{i+1})F(s_i) + \int_{s_i}^{t_{i+1}} F(t)dt)\right\} \end{aligned} \tag{4}$$

# 3 Consecutive improvement method

The main idea of consecutive improvement method proposed in this paper is to check for any possibilities that can help to reduce the total cost. It is noted that the total cost function derived in (4) can be determined if we know the values of $t_i$'s and $s_i$'s. In this research, the values of $t_i$'s will be determined based on the proposed heuristic technique of Wang [14] for the case when shortages are not allowed. Then, for each replenishment cycle $i$, except the last one, $s_i$ will be determined in such a way that the reduction cost associated with cycle $i$ is maximized. The details on the determination of $t_i$'s and $s_i$'s are presented in Section 3.1 and 3.2 below.

## 3.1 Procedure to find $\{t_i\}$

In this section, we briefly review the heuristic technique proposed by Wang [14] to determine replenishment times when shortage is not allowed. For a planning horizon of length $H$, if only one replenishment is used, the order size $Q$ will be the cumulative demand for period $[0,H]$, and the behavior of the inventory level function can be presented as in Fig. 2a.

If another replenishment is used at time $t_2$ ($0 < t_2 < H$), the order sizes of the two replenishments, i.e., $Q_1$ and $Q_2$, will be determined as the cumulative demands in the periods $[0,t_2]$ and $[t_2,H]$, respectively. It is noted that the use of the second replenishment will help to reduce the inventory holding cost but at the price of an additional order (see Fig. 2b.). It is, therefore, easily to see that the two-replenishment policy will be preferred over the one-replenishment policy if the reduction in inventory holding cost is greater than the ordering cost. The optimal value of $t_2$, if exists, can be found by maximizing the difference between the holding cost reduction and the ordering cost.

The same technique can then be applied for the two replenishment cycle, i.e., $[0,t_2]$ and $[t_2,H]$, to detect if other replenishment times should be used. Details about the step-

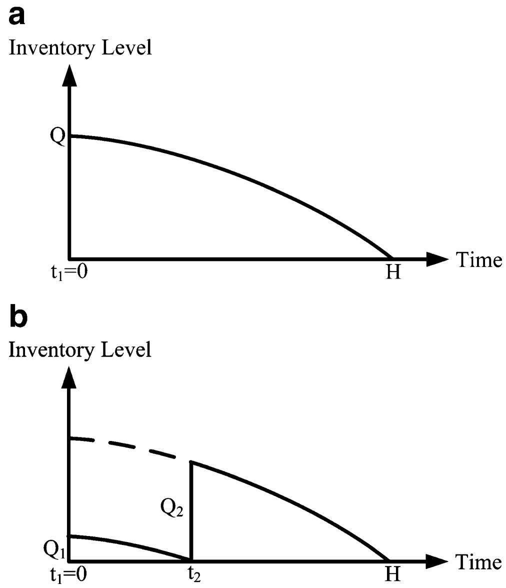


**Fig. 2** **a** Inventory level with one replenishment. **b** Inventory level with two replenishments

by-step procedure to determine $t_i$'s can be referred in the paper of Wang [14] and will not be discussed further here.

3.2 Procedure to find $\{s_i\}$

Once the replenishment times $t_i$'s have been defined, the shortage starting point $s_i$'s in each cycle, except for the last

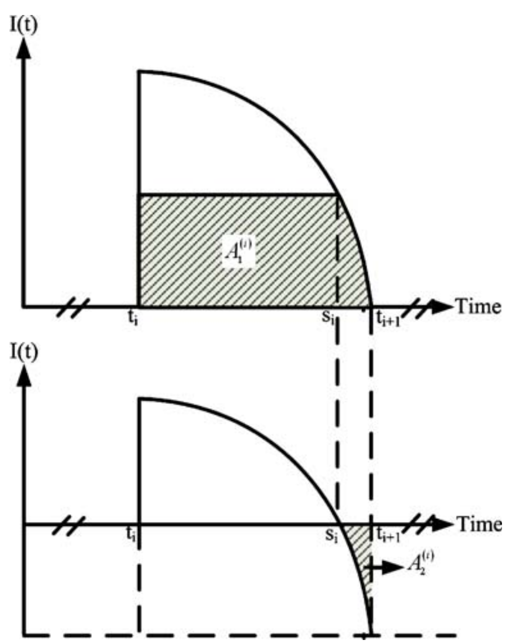


**Fig. 3** Reduction of cumulative holding inventory and cumulative shortage

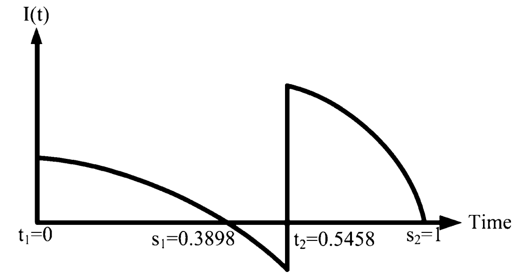


**Fig. 4** Optimal replenishment schedule—a simple sample problem

one, will be determined by solving a maximization problem, in which the objective function is the reduction cost, which is defined as the difference between the reduction in holding cost when shortage is allowed and the shortage cost that is incurred. The above discussed maximization problem can be formulated as follows:

$$\text{Maximize } RC^{(i)} = c_2\left(A_1^{(i)}\right) - c_3\left(A_2^{(i)}\right) \quad (5)$$

$$\text{Subject to} \qquad t_i \leq s_i \leq t_{i+1}$$

in which

$RC^{(i)}$ reduction cost in cycle $i$

$A_1^{(i)}, A_2^{(i)}$ the areas shown in Fig. 3, which represent for the reduction of cumulative inventory and cumulative shortage in cycle $i$

It is noted that, if the maximum reduction cost is positive, the shortage will be allowed in the corresponding

**Table 1** Parameters of the 12 sample problems

| Number | $a$ | $b$ | $c$ | $H$ | $c_1$ | $c_2$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 900 | 100 | 1 | 9 | 2 |
| 2 | 0 | 900 | 100 | 2 | 9 | 2 |
| 3 | 0 | 100 | 5 | 4 | 100 | 2 |
| 4 | 0 | 1,600 | 100 | 3 | 42 | 0.56 |
| 5 | 6 | 1 | 0.005 | 11 | 30 | 1 |
| 6 | 6 | 1 | 0.005 | 11 | 50 | 1 |
| 7 | 6 | 1 | 0.005 | 11 | 60 | 1 |
| 8 | 6 | 1 | 0.005 | 11 | 70 | 1 |
| 9 | 6 | 1 | 0.005 | 11 | 90 | 1 |
| 10 | 100 | 150 | 10 | 1 | 30 | 2 |
| 11 | 100 | 150 | 10 | 1.5 | 30 | 2 |
| 12 | 100 | 150 | 10 | 2 | 30 | 2 |

**Table 2** Total cost of the 12 sample problems resulted from the proposed method

| Problem no. | Total cost | | | | | |
|---|---|---|---|---|---|---|
| | Shortage is allowed | | | | | No shortage |
| | $c_3$=2.5$c_2$ | $c_3$=5$c_2$ | $c_3$=7.5$c_2$ | $c_3$=75$c_2$ | $c_3$=500,000$c_2$ | |
| 1 | 114.791 | 121.035 | 123.654 | 129.329 | 130.053 | 130.053 |
| 2 | 328.689 | 350.861 | 360.088 | 379.944 | 382.462 | 382.462 |
| 3 | 1,119.707 | 1,166.415 | 1,186.014 | 1,228.505 | 1,233.924 | 1,233.925 |
| 4 | 922.432 | 978.567 | 1,002.142 | 1,053.293 | 1,059.820 | 1,059.800 |
| 5 | 260.330 | 277.109 | 284.123 | 299.280 | 301.208 | 301.209 |
| 6 | 340.330 | 357.109 | 364.123 | 379.280 | 381.208 | 381.209 |
| 7 | 380.330 | 397.109 | 404.123 | 419.280 | 421.208 | 421.209 |
| 8 | 420.330 | 437.109 | 444.123 | 459.280 | 461.208 | 461.209 |
| 9 | 500.330 | 517.109 | 524.123 | 539.280 | 541.208 | 541.209 |
| 10 | 139.870 | 145.957 | 148.535 | 154.168 | 154.891 | 154.891 |
| 11 | 217.987 | 229.775 | 234.714 | 245.408 | 246.771 | 246.771 |
| 12 | 337.138 | 351.069 | 356.862 | 369.325 | 370.905 | 370.906 |

cycle; otherwise, the shortage is not allowed because the negative value of maximum reduction cost means that the existence of shortage will make the total relevant cost increases.

The explicit expression for $RC^{(i)}$ can be derived as follows:

$$\begin{aligned}
RC^{(i)} &= c_2A_1^{(i)} - c_3A_2^{(i)} \\
&= c_2\left\{(s_i - t_i)\int_{s_i}^{t_{i+1}} f(t)dt + \int_{s_i}^{t_{i+1}} I(t)dt\right\} - c_3\left\{\int_{s_i}^{t_{i+1}} S(t)dt\right\} \\
&= c_2\left\{(s_i - t_i)\int_{s_i}^{t_{i+1}} f(t)dt + \int_{s_i}^{t_{i+1}} (t - s_i)f(t)dt\right\} \\
&\quad - c_3\left\{\int_{s_i}^{t_{i+1}} (t_{i+1} - t)f(t)dt\right\} \\
&= c_2\left\{\int_{s_i}^{t_{i+1}} (t - t_i)f(t)dt\right\} - c_3\left\{\int_{s_i}^{t_{i+1}} (t_{i+1} - t)f(t)dt\right\} \\
&= c_2\left\{(t_{i+1} - t_i)F(t_{i+1}) + (t_i - s_i)F(s_i) - \int_{s_i}^{t_{i+1}} F(t)dt\right\} \\
&\quad -c_3\left\{(s_i - t_{i+1})F(s_i) + \int_{s_i}^{t_{i+1}} F(t)dt\right\}
\end{aligned} \tag{6}$$

**Proposition** *For each cycle i, the optimal value of $s_i$ always exist and it can be determined as*:

$$s_i^* = \frac{c_2t_i + c_3t_{i+1}}{c_2 + c_3} \tag{7}$$

*Proof* We have

$$\begin{aligned}
\frac{dRC^{(i)}}{ds_i} &= 0 \\
&\Leftrightarrow c_2\{0 + t_if(s_i) - F(s_i) - s_if(s_i) + F(s_i)\} \\
&\quad - c_3\{F(s_i) + s_if(s_i) - t_{i+1}f(s_i) - F(s_i)\} = 0 \\
&\Leftrightarrow c_2t_if(s_i) - c_2s_if(s_i) - c_3s_if(s_i) + c_3t_{i+1}f(s_i) = 0 \\
&\Leftrightarrow -s_if(s_i)[c_2 + c_3] + f(s_i)[c_2t_i + c_3t_{i+1}] = 0 \\
&\Leftrightarrow f(s_i)[c_2t_i + c_3t_{i+1} - (c_2 + c_3)s_i] = 0 \\
&\Leftrightarrow s_i^* = \frac{c_2t_i+c_3t_{i+1}}{c_2+c_3}
\end{aligned}$$

It is noted that the value of $s_i$ defined above is valid, i.e., $t_i < s_i < t_{i+1}$. And it is also noted that when $s_i > s_i^*$, $\frac{dRC^{(i)}}{ds_i} < 0$, and $\frac{dRC^{(i)}}{ds_i} > 0$ when $s_i < s_i^*$. Therefore, $RC^{(i)}$ will be maximized at $s_i^*$.

**Table 3** Optimal replenishment schedule for problem no.1 of the 12 sample problems

| Cycle $i$ | $t_i$ | $s_i$ | Order quantity |
|---|---|---|---|
| 1 | 0.0000 | 0.1396 | 8.8580 |
| 2 | 0.1954 | 0.2969 | 31.6876 |
| 3 | 0.3375 | 0.4301 | 45.3538 |
| 4 | 0.4671 | 0.5491 | 55.2927 |
| 5 | 0.5819 | 0.6643 | 67.1941 |
| 6 | 0.6973 | 0.7735 | 76.3044 |
| 7 | 0.8040 | 0.8760 | 83.0145 |
| 8 | 0.9048 | | 115.6282 |
| Total | | | 483.3333 |
| Ending time | 1.0000 | | |
| Total cost | | | 114.7910 |

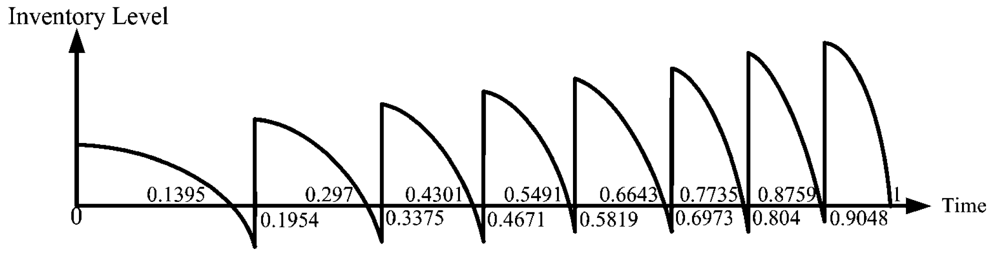


**Fig. 5** Optimal replenishment schedule for problem no.1 of the 12 sample problems

## 4 Numerical experiments

In this section, a simple numerical example will firstly be carried out to illustrate how the proposed technique works. The inputs of the example are as follows: $f(t) = 100 + 150t + 10t^2$, $H = 1$, $c_1 = 30$, $c_2 = 2$, $c_3 = 5$. The solution for this problem will be determined through a two-stage procedure as presented below.

Stage 1 Apply the procedure discussed in Section 3.1 to find the replenishment times $\{t_i\}$ in the planning horizon. The details are as follows:

- At the beginning, only one replenishment at time $t_1 = 0$ will be considered. The order size $Q_1^0$ will be determined as the cumulative demand of the whole planning horizon, and the result is $Q_1^0 = 178.33$.
- In the first iteration, an additional replenishment at time $t_2 \in (0, H)$ will be considered. The optimal value of $t_2$ can be found by maximizing the difference between the holding cost reduction and the ordering cost. The result is $t_2^* = 0.5458$, and this value of $t_2^*$ is acceptable due to the fact that the reduction in holding cost, which is 55.054, is greater than the ordering cost. The existence of $t_2^*$ will form two replenishment cycles in the planning horizon, i.e., $\left[0, t_2^*\right]$ and $\left[t_2^*, H\right]$.

Next iteration will be carried out in each of the above two replenishment cycles to determine if additional replenishment times $t_3 \in \left(0, t_2^*\right)$ and $t_4 \in \left(t_2^*, H\right)$ will help to reduce total inventory cost further. The same procedure is employed to find the optimal values of $t_3$ and $t_4$, and the results are $t_3^* = 0.2906$ and $t_4^* = 0.7818$, respectively. However, the inclusion of both $t_3^*$ and $t_4^*$ will not help to reduce the total inventory cost due to the fact that the holding cost reductions in both cases, which are 12.199 and 12.443, are less than the cost of an additional order.

As no further improvement in total inventory cost can be found, we stop the iteration process in this stage and go on to the stage 2. It is noted that the whole planning horizon has been divided into two replenishment cycles with the associated replenishment times $t_1 = 0$ and $t_2 = 0.5458$.

Stage 2 In this stage, the procedure proposed in Section 3.2 will be employed to determine the optimal shortage starting point in the first replenishment cycle defined in stage 1, and the results is $s_1^* = 0.3898$. It is noted that no shortage will be allowed in the second replenishment cycle because this is the last cycle of the planning horizon. The solution has been found, i.e., $t_1 = 0$, $t_2 = 0.5458$, and $s_1 = 0.3898$, and illustrated in Fig. 4. The associated order sizes and total cost are $Q_1 = 50.5770$, $Q_2 = 100.8777$, and $C = 139.8699$.

To further illustrate the applicability of the proposed method, the 12 sample problems taken from Yang et al. [13] will be considered in the remaining part of this section. In these problems, the demand rate has the form $f(t) = a +$

**Table 4** Comparison with the proposed technique of Chen et al. [24]

| Case | Our proposed technique | | | Nelder-Mead Algorithm (Chen et al. [24]) | | |
|---|---|---|---|---|---|---|
| | $n$ | Total cost | Computational time[a] (second) | $n$ | Total cost | Computational time[a] (second) |
| $c_3$=2.5$c_2$ | 22 | 328.6894 | 0.188 | 15 | 326.5964 | 290.703 |
| $c_3$=5$c_2$ | 22 | 350.8613 | 0.109 | 17 | 353.6083 | 276.468 |
| $c_3$=7.5$c_2$ | 22 | 360.0878 | 0.172 | 16 | 368.2256 | 410.359 |
| $c_3$=75$c_2$ | 22 | 379.9438 | 0.141 | 20 | 379.5257 | 556.671 |

[a] Recorded on a PC with Intel P4 3.4 GHz and 1 GB RAM

$bt + ct^2$ ($a,b,c \geq 0$) with the detailed parameters given in Table 1.

The results of the optimal total costs for the sample problems with various values of shortage cost $c_3$ are shown Table 2. Results of the non-shortage policy, which are determined from the algorithm of Wang [14], are also presented in Table 2 for comparison purpose. From the results, it can be seen that the total cost that resulted from the policy that allows shortage occurrence is always better than the total cost of the policy without shortage allowance. In addition, it can be observed that, when the shortage cost $c_3$ increases, the total cost of the policy with shortage allowance will increase and approach the total cost of the policy without shortage allowance.

For more details on the performance of the proposed method, the solution of Problem 1 with $c_3 = 2.5c_2$ are presented in Table 3 and also illustrated on Fig. 5.

From the result presented in Table 3, it can be seen that eight replenishments will be needed, and the associated total cost is 114.7910, which is lower than the total cost reported by Wang [14] for the situation when shortages are not allowed.

To further illustrate the applicability of our proposed heuristic technique, comparison with the proposed technique of Chen et al. [24], which is based on the Nelder–Mead algorithm, will be conducted. Related to the research of Chen et al. [24], their numerical results are for the case of the whole product life cycle, and hence, it is inappropriate to compare the performance of our proposed technique with their technique in this situation due to the fact that our technique aims at the increasing demand pattern situation only. However, because the Nelder–Mead algorithm is a general search technique, the proposed technique of Chen et al. [24] can be applied also for the case of increasing demand pattern. Therefore, for comparison purpose, the second sample problem taken from Yang et al. [13] is solved by using both techniques, and the results are presented below in Table 4

From the results in Table 4, it can be seen that our proposed technique is comparable with the proposed technique of Chen et al. [24] in terms of solution quality. Moreover, the computational times required in our technique are much smaller.

## 5 Conclusion

The research presented in this paper can be considered as an extension of the work of Wang [14]. The technique developed in this paper gives a new way to solve the inventory problem with nonlinear increasing demand pattern, which takes into consideration the allowance of shortage. From the results of numerical experiments conducted in this research, it can be seen that the total inventory cost that resulted from the policy that allows shortage is always lower than the total inventory cost of the policy without shortage if holding cost is significant as compared to shortage cost. Besides the economic advantage, it should be also noted that the allowance of shortage in the inventory policy can also help to reduce the pressure on warehouse allocation in case of space shortage.

The main contribution of this research is the new concept of reduction cost used in the proposed solution technique, which is easy for practitioners to understand and employ. Moreover, the proposed technique can always guarantee that the total length of replenishment intervals always match exactly with the planning horizon. This requirement has not been satisfied in the former researches of Yang et al. [22, 23]. However, it should be noted that the technique proposed in this paper can only help to deal with the cases where shortages are not allowed at the beginning of each replenishment cycle. For future research directions, the research conducted in this paper should be extended to deal with the cases at which shortages are allowed to take place at the beginning of the replenishment cycle or when lost sales policy is considered.

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