A repetitive forward rolling technique for inventory policy with non-linear increasing demand pattern considering shortage

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Abstract: Demand of any particular products might not be stable, e.g., in the growth stage of the product life cycle where demand of most products might possess increasing functional form. The famous EOQ model is, then, not appropriate in this situation, since it was developed under the assumption of constant demand pattern. The research in this paper is focused on inventory decision problem with non-linear increasing demand pattern and considering shortage, by proposing a heuristic method based on repetitive forward rolling technique for determining the inventory policy for this case, i.e., the replenishment times and shortage points. The proposed technique is developed in such a way that demands during a predefined planning horizon are exactly met. Numerical experiments that are conducted to illustrate the applicability of the proposed technique show that it can provide competitive results when it is compared with other proposed techniques in the past researches.

Keywords: inventory; non-linear demand pattern; increasing demand rate; heuristic; repetitive forward rolling technique.

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1 Introduction

The use of stable demand assumption in the development of inventory policy sometimes is not always proper, since in reality, demands for most products are not always stable during the whole life cycle. Based on the concept of product life cycle (Cox, 1967; Golder and Tellis, 2004), every product has a series of stages which are introduction, growth, mature and decline, in which the growth stage is usually characterised by increasing demand pattern.

The most well-known inventory policy, EOQ, was developed by assuming that demand rate is constant, therefore the use of EOQ model for the case of increasing demand pattern is inappropriate. A proper inventory policy is then needed in this situation.

According to Diponegoro and Sarker (2002), the life cycle of product is finite, therefore the development of inventory policy in each stage of the product life cycle should consider finite planning horizon. Many researches have been conducted for the development of inventory policy for the case of linear increasing demand pattern under finite planning horizon. The pioneers were Resh et al. (1976) and Donaldson (1977) who provided the exact solution procedure. Further researches in the development of exact solution procedure were done by Hariga (1993), Lo et al. (2002) and Ritchie (1984).

Besides exact solution procedures, many researchers also developed heuristic procedures for determining inventory policy under linear increasing demand pattern, such as Silver (1979), Phelps (1980), Teng (1994) and Mitra et al. (1984). In general, those heuristic procedures can provide the solutions in which the total inventory costs were only slightly higher than that of the exact solution procedure provided by Donaldson (1977), except for the method of Teng (1994), which provided the same result as Donaldson's.

It is noted that for some products, it is fine to use the assumption of linear increasing demand form, however for many other products; a non-linear form will be a better representation of demand pattern. The development of inventory policy for the case of non-linear increasing demand pattern were first done by Yang et al. (1999) and Wang (2002) who developed analytic eclectic algorithm and consecutive method, respectively. In those two researches, the inventory policies were developed for the systems in which no shortages are allowed. In many practical inventory systems, shortage might occur when customer's orders arrive but there are no stocks of goods available. Shortage sometimes is economically preferable, i.e., when holding cost is significant enough as compared with shortage cost. It is noted that shortage can be either partially or completely backlogged. According to Zhou et al. (2004), inventory model considering shortage can be grouped into two categories which are:

1 inventory followed by shortage (IFS) in which every cycle starts with replenishment and end with shortages (except in last cycle)

2 shortage followed by inventory (SFI) in which every cycle starts with shortage before replenishment is received.

These two categories were also discussed by Teng et al. (1997) who further divided inventory models considering shortage into four. The first two models (Models 1 and 2) are IFS policies in which Model 1 does not permit shortage in the last cycle while Model 2 does. The last two models (Models 3 and 4) are SFI policies in which shortages are allowed in the last cycle of Model 4 only.

The research on inventory policy for linear increasing demand pattern considering shortage, to the best of author's knowledge, was initially done by Deb and Chaudhuri (1987). They developed a heuristic procedure for inventory model that is similar with Model 2 of Teng et al. (1997) by assuming that the shortage period in each cycle is a constant fraction of the cycle length. This work can be considered as the extension of the work of Silver (1979) in which shortage is taken into consideration. Later, Dave (1989a) developed a heuristic procedure for Model 1 of Teng et al. (1997) and the total inventory cost resulted is close to the optimal solution provided by Dave (1989b).

Other researches on inventory policy for linear increasing demand considering shortage were done by Dave (1989b), Murdeshwar (1988), Hariga (1994) and Goyal et al. (1996) who developed the optimal solution. It is noted that both Dave (1989b) and Murdeshwar (1988) worked on IFS policy under the condition that shortages are not allowed to happen in the last cycle. Hariga (1994) also applied IFS policy, but shortages are allowed to occur in the last cycle. In addition, the model he developed is used not only for linear increasing but also for non-linear increasing and decreasing demand pattern. It is also noted that the research conducted by Goyal et al. (1996) dealt with both IFS and SFI policies.

While the research on developing exact method for IFS policy with non-linear increasing demand has been done by Hariga (1994), the research on SFI policy for the case of positive non-linear increasing demand was done by Yang (2006) who developed exact solution procedure based on backward recursive algorithm. From numerical results, Yang (2006) also concluded that Model 4 of Teng et al. (1997) seems to give the best result among the four models.

Another heuristic method for SFI policy for the case of non-linear increasing demand was also done by Yang et al. (2002) by using the forward recursive algorithm. In their research, the occurrence of shortages is allowed in the last cycle of planning horizon. It is noted that even though shortage is assumed to be completely backlogged but both Yang et al. (2002) and Yang (2006) have not mentioned yet on how to handle the shortage demand at the end of planning horizon.

A recent development was done by Chen et al. (2007) who proposed a direct search method based on Nelder-Mead algorithm to find the replenishment times. In their research, Chen et al. (2007) considered the whole product life cycle by assuming that demand rate follows a revised beta function. Astanti and Luong (2009) extended the idea of consecutive improvement method (Wang, 2002) into a new heuristic technique for finding the replenishment times and shortage starting points.

Current researches have been conducted by some researchers in the area of inventory policy problem for non-linear increasing demand pattern by considering more characteristics of inventory policy problem in to the model such as partial backlog case, deterioration rate for perishable product, and time value of money. However, solution methodology are developed under the assumption of the very specific non-linear increasing demand pattern such as quadratic demand pattern (Panda et al., 2009a, 2009b; Sarkar et al., 2010), polynomial demand pattern (Bai and Kendall, 2008), exponential demand pattern (Wu, 2002), and ramp-type demand pattern (Manna and Chiang, 2010; Kawakatsu, 2010; Roy and Chaudhuri, 2011).

Based on the literature review above, it can be seen that the exact solution procedure for the case of non-linear increasing demand pattern considering shortage has been conducted by Hariga (1994) who worked on IFS policy, under the condition that it still remains shortage in the last cycle, which means that demand over the pre-established planning horizon will not be met. Yang (2006) also developed exact solution for both IFS and SFI policies, however, it requires a lot of mathematical effort and the method is limited to a very specific demand pattern. Therefore, the development of method to solve inventory replenishment problem that is simple in the mathematical development and can be applied in more general demand pattern is needed. This research will then be focused on this issue, by taken into consideration the IFS policy with non-linear increasing demand pattern. It is noted that even though the inventory policy problem addressed in this paper is not yet considering more characteristics of inventory policy problem such as partial backlog case, deterioration rate for perishable product, and time value of money, however, the solution methodology that is developed in this paper can be applied for general non-linear increasing demand. In addition, shortages are not allowed to occur in the last cycle to ensure that total demand over the pre-established planning horizon can always be met, i.e., shortages are completely backlogged. In this paper, at first, a method to find intermediate replenishment times at fixed values of shortage points, and to find intermediate shortage points at fixed value of replenishment times is proposed. Then, a proposed repetitive forward rolling technique will be performed to adjust the replenishment times and shortage points so that total inventory cost can be reduced. This procedure will be performed until no improvement in total cost is realised. The performance of the proposed technique then will be tested by comparing its result with the results from Yang's (2006) technique and the technique based on Nelder-Mead algorithm developed by Chen et al. (2007).

The remaining parts of this paper are organised as follows. Section 2 presents problem definition and the mathematical model in which the expression of total inventory cost function will be derived. In Section 3, the procedures to find intermediate replenishment times and intermediate shortage points are explained. The development of the repetitive forward rolling technique is explained in Section 4. Numerical experiments to illustrate the applicability of the proposed technique are then presented in Section 5. Then, some discussion about the proposed method are presented in Section 6. Finally, some concluding remarks and further work will be discussed in Section 7.

2 Problem definition and mathematical model

The following notation will be used throughout the paper:

- H is the length of the planning horizon under consideration
- $f(t)$ is the instantaneous demand rate at time t (general non-linear increasing function)
- c_1 is the ordering cost per order
- c_2 is the holding cost per unit per unit time
- c_3 is the shortage cost per unit per unit time
- n is the number of replenishments in the planning horizon
- t_i is the i^{th} replenishment time ($i = 1, 2,\ldots,n$)
- s_i is the shortage starting point of cycle i, which is the time at which the inventory level reaches zero in the i^{th} cycle $[t_i, t_{i+1}]$; ($i = 1,2,\ldots,n - 1$)
- $I(t)$ is the inventory level at time t which is evaluated after the replenishment arrives at time t_i and before the replenishment arrives at time t_{i+1} in the i^{th} cycle $[t_i, t_{i+1}]$.

The behaviour of the inventory level function of the inventory problem dealt with in this paper is presented in Figure 1. The problem considered in this paper is using the following assumptions:

a demand rate posses a non-linear increasing pattern

b the replenishment is made at time t_i ($i = 1, 2,\ldots,n$) in which $t_1 = 0$

c lead time is negligible, i.e., replenishment is instantaneous

d the quantity received at t_i is used partly to meet accumulated shortages in the previous cycle from time s_{i-1} to t_i ($s_{i-1} < t_i$); ($i = 2,\ldots,n$)

e no shortages at the beginning ($t_1 = 0$) and the end of the planning horizon ($s_n = H$)

Figure 1 Inventory level over the whole planning horizon

The optimisation problem appeared here is to find the optimal number of cycle (n), the shortage starting point of each cycle (s_i) and the replenishment time of each cycle (t_i) in order to minimise the total relevant cost (C), which includes ordering cost, holding cost, and shortage cost. Using the assumptions above, the expression for the total relevant cost of the inventory system during the planning horizon H when n orders are placed is expressed as follows:

$$C\left(n,\{s_i\},\{t_i\}\right)=nc_1+c_2\sum_{i=1}^{n}I_i+c_3\sum_{i=1}^{n-1}S_i \tag{1}$$

in which

- I_i is the cumulative holding inventory during cycle i
- S_i is the cumulative shortage during cycle i.

For each cycle i from t_i to t_{i+1}, the cumulative holding inventory I_i and cumulative shortage S_i can be derived as in Sections 2.1 and 2.2 below.

2.1 Cumulative holding inventory I_i

If $F(t)$ is denoted as the cumulative demand up to time t ($t_i \le t \le t_{i+1}$), then

$$F(t)=\int_0^t f(t)dt \tag{2}$$

It is also noted that

$$f(t)=\frac{dI(t)}{dt} \text{ and hence, } I(t)=\int_t^{s_i} f(\tau)d\tau \quad \left(t_i \le t \le s_i\right) \tag{3}$$

The cumulative inventory I_i in cycle i can then be determined as:

$$I_i=\int_{t_i}^{s_i} I(t)dt=\int_{t_i}^{s_i}\int_t^{s_i} f(\tau)d\tau dt \tag{4}$$

Hence,

$$I_i=\left(s_i-t_i\right)F\left(s_i\right)-\int_{t_i}^{s_i}F(t)dt \tag{5}$$

2.2 Cumulative shortage S_i

The cumulative shortage S_i in cycle i ($i = 1, 2,\ldots,n - 1$) can be determined as:

$$S_i=\int_{s_i}^{t_{i+1}}\left(-I(t)\right)dt=\int_{s_i}^{t_{i+1}}\int_{s_i}^{t} f(\tau)d\tau dt \tag{6}$$

Hence,

$$S_i=\left(s_i-t_{i+1}\right)F\left(s_i\right)+\int_{s_i}^{t_{i+1}}F(t)dt \tag{7}$$

From (2) and (3), the expression of the total cost can be defined as follows:

$$C\left(n, \{s_i\}, \{t_i\}\right) = nc_1 + c_2 \sum_{i=1}^{n} \left\{ \left(s_i - t_i\right) F\left(s_i\right) - \int_{t_i}^{s_i} F(t)dt \right\} + c_3 \sum_{i=1}^{n-1} \left\{ \left(s_i - t_{i+1}\right) F\left(s_i\right) + \int_{s_i}^{t_{i+1}} F(t)dt \right\} \tag{8}$$

3 Proposed procedure to find replenishment times and intermediate shortage points

The procedure to find the replenishment time t_{i+1} if the value of s_i and s_{i+1} are known, and intermediate shortage point s_i if the value of t_i and t_{i+1} are known will be explained in detail in Section 3.1 and 3.2 below.

3.1 Procedure to find intermediate replenishment points $\{t_{i+1}\}$

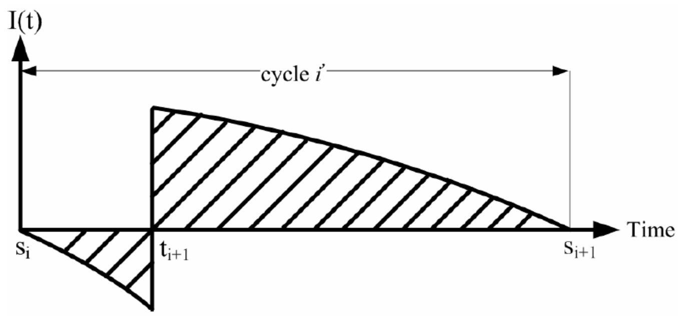
Consider cycle i' which is defined as the interval $[s_i, s_{i+1}]$ in which s_i and s_{i+1} are fixed. Then replenishment time t_{i+1} has to be obtained so that the total cost in the corresponding cycle is minimised (see Figure 2 for illustration). Total cost for cycle i' by locating replenishment time at t_{i+1}, denoted by TCS_i, is determined below.

$$TCS_i = c_1 + c_3 \left\{ \int_{s_i}^{t_{i+1}} \int_{s_i}^{t} f(\tau)d\tau dt \right\} + c_2 \left\{ \int_{t_{i+1}}^{s_{i+1}} \int_{t}^{s_{i+1}} f(\tau)d\tau dt \right\} \tag{9}$$

or it can be expressed as:

$$TCS_i = c_1 + c_3 \left\{ \left(s_i - t_{i+1}\right) F\left(s_i\right) + \int_{s_i}^{t_{i+1}} F(t)dt \right\} + c_2 \left\{ \left(s_{i+1} - t_i\right) F\left(s_{i+1}\right) - \int_{t_{i+1}}^{s_{i+1}} F(t)dt \right\} \tag{10}$$

Figure 2 Inventory level for cycle i'



The optimal value of t_{i+1} (if exist) is the solution of the following equation:

$$\frac{\partial TCS_i}{\partial t_{i+1}} = 0 \tag{11}$$

or equivalently,

$$c_3 \left\{ -F\left(s_i\right) + F\left(t_{i+1}\right) \right\} + c_2 \left\{ -F\left(s_{i+1}\right) + F\left(t_{i+1}\right) \right\} = 0 \tag{12}$$

or

$$F\left(t_{i+1}\right) = \frac{c_2 F\left(s_{i+1}\right) + c_3 F\left(s_i\right)}{c_2 + c_3} \tag{13}$$

Since the values of s_{i+1} and s_i are known and F is a strictly increasing function, therefore the value of t_{i+1}^* that satisfies equation (13) can be uniquely determined by using any search algorithm, e.g., bisection algorithm.

To prove that the unique solution t_{i+1}^* from equation (13) is optimal it has to be proven that the second derivative of the total cost TCS_i with respect to replenishment time t_{i+1} is greater than zero at t_{i+1}^*. In fact,

$$\left.\frac{\partial TCS_i}{\partial t_{i+1}^2}\right|_{t_{i+1}=t_{i+1}^*} = \left(c_2 + c_3\right) f\left(t_{i+1}^*\right) > 0 \tag{14}$$

From inequality (14) it can be seen that $\left.\frac{\partial TCS_i}{\partial t_{i+1}^2}\right|_{t_{i+1}=t_{i+1}^*} > 0$. This implies that TCS_i reaches its minimum at t_{i+1}^*.

3.2 *Procedure to find intermediate shortage points $\{s_i\}$*

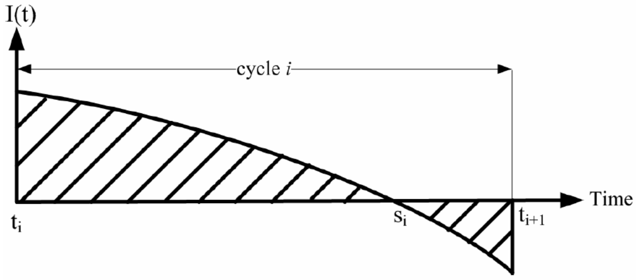
Consider cycle i which is defined as the interval $[t_i, t_{i+1}]$ in which t_i and t_{i+1} are fixed. The shortage point s_i is obtained so that total cost in the corresponding cycle is minimised (see Figure 3 for illustration). Total cost for cycle i, denoted by TCI_i, is then determined as follows.

$$TCI_i = c_1 + c_2 \left\{ \int_{t_i}^{s_i} \int_{t}^{s_i} f(\tau) d\tau dt \right\} + c_3 \left\{ \int_{s_i}^{t_{i+1}} \int_{s_i}^{t} f(\tau) d\tau dt \right\} \tag{15}$$

or it can be expressed as:

$$TCI_i = c_1 + c_2 \left\{ \left(s_i - t_i\right) F\left(s_i\right) - \int_{t_i}^{s_i} F(t) dt \right\} + c_3 \left\{ \left(s_i - t_{i+1}\right) F\left(s_i\right) + \int_{s_i}^{t_{i+1}} F(t) dt \right\} \tag{16}$$

Figure 3 Inventory level for cycle *i*



The optimal value of s_i (if exist) is the solution of the following equations:

$$\frac{\partial TCI_i}{\partial s_i} = 0 \tag{17}$$

or equivalently,

$$c_2\left(s_i - t_i\right) f\left(s_i\right) + c_3\left(s_i - t_{i+1}\right) f\left(s_i\right) = 0 \tag{18}$$

The last expression then can be written as:

$$\left(c_2 + c_3\right) s_i^* - c_2 t_i - c_3 t_{i+1} = 0 \tag{19}$$

or it can be simplified as:

$$s_i = \frac{c_2 t_i + c_3 t_{i+1}}{c_2 + c_3} \tag{20}$$

By using equation (20) the unique solution of shortage point s_i^* then can be determined.

To prove that the unique solution s_i^* from equation (20) is optimal it has to be proven that the second derivative of total cost TCI_i with respect to shortage point s_i is greater than zero at s_i^*. In fact, the expression for the second derivative is

$$\left.\frac{\partial TCI_i}{\partial s_i^2}\right|_{s_i = s_i^*} = \left(c_2 + c_3\right) f\left(s_i^*\right) + f'\left(s_i^*\right)\left(\left(c_2 + c_3\right) s_i^* - c_2 t_i - c_3 t_{i+1}\right) \tag{21}$$

It is noted that at s_i^*, $((c_2 + c_3))s_i^* - c_2 t_i - c_3 t_{i+1}) = 0$.

Hence, $\left.\frac{\partial TCI_i}{\partial s_i^2}\right|_{s_i = s_i^*} = \left(c_2 + c_3\right) f\left(s_i^*\right) > 0.$ This implies that total cost TCI_i reaches its minimum at s_i^*.

4 Repetitive forward rolling technique

In this section, a repetitive forward rolling technique is proposed to help adjust the predefined replenishment times $\{t_{i+1}\}$ (i = 1, 2,…,n – 1) and shortage points $\{s_i\}$

($i = 1, 2, \ldots, n - 1$) for the planning horizon of length H, so that total cost can be gradually reduced. The proposed procedure is as follows:

Step 1 Divide the planning horizon of length H into n equal cycles in which cycle i goes from t_i to t_{i+1} ($i = 1, 2, \ldots, n - 1$; $t_1 = 0$), except in the last cycle, the length of cycle n goes from t_n to s_n ($s_n = H$). Assign a large value for TC = Inf.

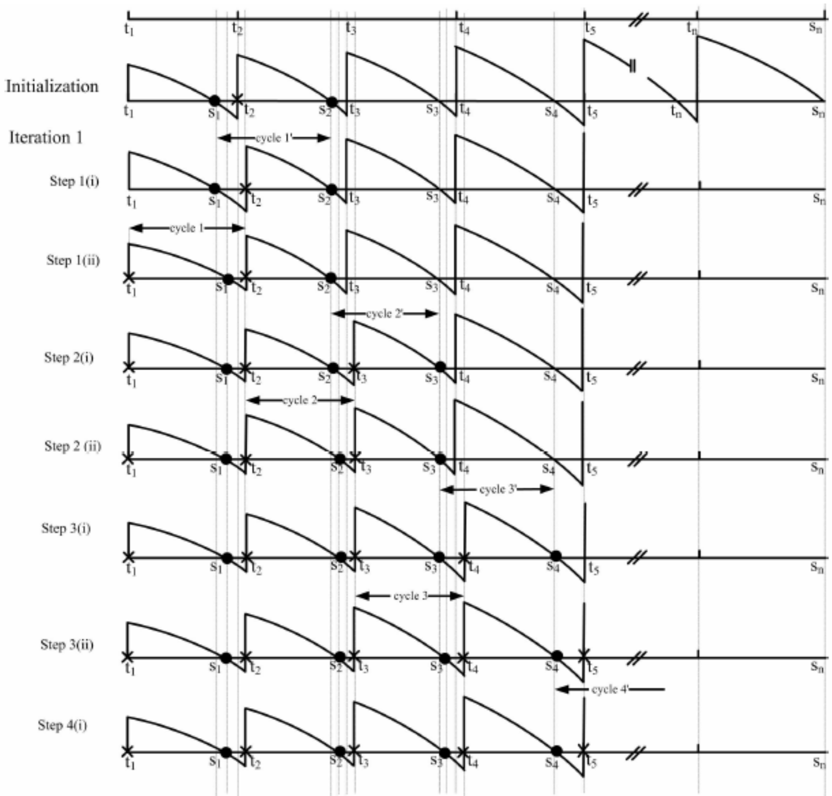
Step 2 Find shortage point s_i in each cycle i ($i = 1, 2, \ldots, n - 1$), by using equation (20).

Step 3 Repetitive forward rolling technique

- a Set $i = 1$.
- b Do.
  - 1 Consider cycle i' $[s_i, s_{i+1}]$, apply equation (13) and then update the value of t_{i+1}.
  - 2 Consider cycle i $[t_i, t_{i+1}]$, apply equation (20) and then update value of s_i. Go to the next step.
- c Update $i = i + 1$. If $i < n$, return to Step 3.b. Otherwise, go to Step 3.d.
- d Compute total cost TC and check if the total cost function is reduced. If yes, repeat Step 3; otherwise, stop the iteration procedure.

The proposed repetitive forward rolling procedure is illustrated in Figure 4.

Figure 4 Repetitive forward rolling procedure



In order to prove that the repetitive forward rolling procedure is converge, it has to be proven that the replenishment time t_i and shortage point s_i are increasing series, as it is explained below:

1 Consider the first iteration of the repetitive forward rolling procedure

- At the Step 1(1), it is found that:

  $$F\left(t_2^{(1)}\right)=\frac{c_2F\left(s_2^{(0)}\right)+c_3F\left(s_1^{(0)}\right)}{c_2+c_3} \tag{22}$$

  with:

  $$s_1^{(0)}=\frac{c_3t_2^{(0)}+c_2t_1^{(0)}}{c_2+c_3}=(1-\lambda)t_2^{(0)}+\lambda t_1^{(0)} \tag{23}$$

  $$s_2^{(0)}=\frac{c_3t_3^{(0)}+c_2t_2^{(0)}}{c_2+c_3}=(1-\lambda)t_3^{(0)}+\lambda t_2^{(0)} \tag{24}$$

  where

  $$\lambda=\frac{c_2}{c_2+c_3} \tag{25}$$

  As if *f*(*t*) is increasing function, then *F*(*t*) is strictly convex function, therefore:

  $$\begin{aligned}F\left(t_2^{(1)}\right)&=\lambda F\left[(1-\lambda)t_3^{(0)}+\lambda t_2^{(0)}\right]+(1-\lambda)F\left[(1-\lambda)t_2^{(0)}+\lambda t_1^{(0)}\right]\\&>F\left[\lambda(1-\lambda)\left(t_3^{(0)}+t_1^{(0)}\right)+\left(\lambda^2+(1-\lambda)^2\right)t_2^{(0)}\right]\end{aligned} \tag{26}$$

  $$F\left(t_2^{(1)}\right)>F\left[2\lambda(1-\lambda)t_2^{(0)}+\left(\lambda^2+(1-\lambda)^2\right)t_2^{(0)}\right] \tag{27}$$

  $$F\left(t_2^{(1)}\right)>F\left(t_2^{(0)}\right) \tag{28}$$

  So: $t_2^{(1)}>t_2^{(0)}$.

- At the Step 1(2): due to: $t_2^{(1)}>t_2^{(0)}$: $s_1^{(1)}>s_1^{(0)}$.
- At the Step 2(1): similarly: $t_3^{(1)}>t_3^{(0)}$.
- At the Step 2(2): due to: $t_2^{(1)}>t_2^{(0)}$ and $t_3^{(1)}>t_3^{(1)}$: $s_2^{(1)}>s_2^{(0)}$.

  So, at the end of the first iteration, all $s_i^{(0)}$ and $t_i^{(0)}$ will be updated to the higher value.

2 Consider the second iteration of the repetitive forward rolling procedure

- At the Step 1(1), it is known that:

  $$F\left(t_2^{(2)}\right)=\frac{c_2F\left(s_2^{(1)}\right)+c_3F\left(s_1^{(1)}\right)}{c_2+c_3} \tag{29}$$

$$F\left(t_2^{(2)}\right) = \lambda F\left(s_2^{(1)}\right) + (1-\lambda)F\left(s_1^{(1)}\right) \quad (30)$$

As $s_1^{(1)} > s_1^{(0)}$ and $s_2^{(1)} > s_2^{(0)}$, therefore $F(s_1^{(1)}) > F(s_1^{(0)})$ and $F(s_2^{(1)}) > F(s_2^{(0)})$.
Hence,

$$\begin{aligned} F\left(t_2^{(2)}\right) &= \lambda F\left(s_2^{(1)}\right) + (1-\lambda)F\left(s_1^{(1)}\right) \\ &> \lambda F\left(s_2^{(0)}\right) + (1-\lambda)F\left(s_1^{(0)}\right) \\ &> F\left(t_2^{(1)}\right) \end{aligned} \quad (31)$$

So: $t_2^{(2)} > t_2^{(1)}$.

- At the Step 1(2): due to: $t_2^{(2)} > t_2^{(1)}$: $s_1^{(2)} > s_1^{(1)}$.
- At the Step 2(1): similarly: $t_3^{(2)} > t_3^{(1)}$.
- At the Step 2(2): due to: $t_2^{(2)} > t_2^{(1)}$ and $t_3^{(2)} > t_3^{(1)}$: $s_2^{(2)} > s_2^{(1)}$.

So, at the end of the second iteration, all $s_i^{(1)}$ and $t_i^{(1)}$ will be updated to the higher value. These prove that replenishment time t_i and shortage point s_i are increasing series.

5 Numerical experiments

To illustrate the applicability of the proposed method for solving inventory policy problem by considering non-linear increasing demand pattern, numerical experiments will be conducted. Two sample set problems are taken into consideration. The first sample set problem is provided by Yang (2006), who solved the problem by using backward recursive algorithm for quadratic demand pattern.

The second sample set problems is the 12 sample problems provided by Yang et al. (1999) for polynomial demand pattern and non-shortage case, however in this paper the 12 sample problems provided by Yang et al. (1999) are modified by incorporating shortage cost.

5.1 Sample problem 1 (Yang, 2006)

Consider the demand function of the form: $f(t) = bt^u$ with $u = 2$, $b = 900$. The other parameters are set as follows: $H = 1$, $c_1 = 4.5$, $c_2 = 1$, $c_3 = 3.5$.

The number of cycles n used in this sample problem is determined based on the formula developed by Teng (1996) for the case of linear increasing demand, where

$$n = \text{rounded integer of } \left\{\left[c_2 c_3 HF(H)\right] \big/ \left[2c_1\left(c_2 + c_3\right)\right]\right\}^{1/2} = 5$$

Sensitivity analysis can be conducted later to find the appropriate value of n. The step-by-step procedure to determine t_{i+1} ($i = 1, 2,\ldots,n - 1$) and s_i ($i = 2,\ldots,n$) is presented below:

Step 1 Determine the initial values of t_i: $t_1 = 0$, $t_2 = 0.2$, $t_3 = 0.4$, $t_4 = 0.6$, $t_5 = 0.8$, and set TC = Inf.

Step 2 Find shortage point s_i in each cycle i ($i = 1, 2, \ldots, n - 1$), by using equation (14): $s_1 = 0.1556$, $s_2 = 0.3556$, $s_3 = 0.5556$, $s_4 = 0.7556$, $s_5 = H = 1$.

Step 3 Repetitive forward repetitive rolling technique

- *Iteration 1(1):* Consider cycle 1′ from $s_1 = 0.1556$ to $s_2 = 0.3556$. Find the new value of t_2 by using equation (8), then the new value of $t_2 = 0.2346$ is obtained. Update the value of t_2, from 0.2 to 0.2346.
- *Iteration 1(2):* Consider cycle 1 from $t_1 = 0$ to $t_2 = 0.2346$. Find the value of s_1 by using equation (14), then the new value of $s_1 = 0.1825$ is obtained. Update the value of s_1, from 0.1556 to 0.1825.
- *Iteration 2(1):* Consider cycle 2′ from $s_2 = 0.3556$ to $s_3 = 0.5556$. Find the new value of t_3 by using equation (8), then the new value of $t_3 = 0.4181$ is obtained. Update the value of t_3, from 0.4 to 0.4181.
- *Iteration 2(2):* Consider cycle 2 from $t_2 = 0.2346$ to $t_3 = 0.4181$. Find the value of s_2 by using equation (14), then the new value of $s_2 = 0.3773$ is obtained. Update the value of s_2, from 0.3556 to 0.3773.
- *Iteration 3(1):* Consider cycle 3′ from $s_3 = 0.5556$ to $s_4 = 0.7556$. Find the new value of t_4 by using equation (8), then the new value of $t_4 = 0.6120$ is obtained. Update the value of t_4, from 0.6 to 0.6120.
- *Iteration 3(2):* Consider cycle 3 from $t_3 = 0.4181$ to $t_4 = 0.6120$. Find the value of s_3 by using equation (14), then the new value of $s_3 = 0.5689$ is obtained. Update the value of s_3, from 0.5556 to 0.5689.
- *Iteration 4(1):* Consider cycle 4′ from $s_4 = 0.7556$ to $s_5 = 1$. Find the new value of t_5 by using equation (8), then the new value of $t_5 = 0.8231$ is obtained. Update the value of t_5, from 0.8 to 0.8231.
- *Iteration 4(2):* Consider cycle 4 from $t_4 = 0.6120$ to $t_5 = 0.8231$. Find the value of s_4 by using equation (14), then the new value of $s_4 = 0.7762$ is obtained. Update the value of s_4, from 0.7556 to 0.7762.

Table 1 Numerical result of sample problem 1

I	*1*	*2*	*3*	*4*	*5*	*Total cost*
			After the first forward step			
t_i^*	0	0.2346	0.4181	0.6120	0.8231	49.0902
s_i^*	0.1825	0.3773	0.5689	0.7762	1	
			After the second forward step			
t_i^*	0	0.2554	0.4384	0.6304	0.8395	47.7546
s_i^*	0.1987	0.3977	0.5878	0.7931	1	
			Final replenishment schedule			
t_i^*	0	0.3964	0.6099	0.7719	0.9076	43.2140
s_i^*	0.3083	0.5625	0.7360	0.8774	1	

Determine total cost *TC* from equation (1): *TC* = 49.0902, and repeat Step 3 until no improvement (i.e., reduction) has been found. For the current sample problem the result from the first forward step and the second one, as well the final replenishment schedules are presented in Table 1.

It is noted that the numbers of replenishment cycles *n* used in the examples are determined from the formula derived by Teng (1996), which is originally developed for the case of linear increasing demand pattern. Therefore, in order to find the best value of *n*, sensitivity analysis should be conducted to investigate the effect of *n* on the total cost function, and the results are presented in Table 2.

Table 2 Sensitivity analysis on the effect of *n* of sample problem 1

I	*n = 4*		*n = 5*		*n = 6*	
	s_i	t_i	s_i	t_i	s_i	t_i
1	0.35162	0	0.3083	0	0.27782	0
2	0.64135	0.45208	0.5625	0.3964	0.50677	0.3572
3	0.83899	0.69543	0.7360	0.6099	0.66304	0.54951
4	1	0.88001	0.8774	0.7719	0.79055	0.69548
5			1.0000	0.9076	0.90109	0.81771
6					1	0.92492
TC	44.9502		*43.214*		43.8076	

Based on the result above, it can be seen that the pre-selected values of *n* used in the sample problem 1 is also the best value in this case.

5.2 Sample set problem 2

In this part, the 12 sample problems of Yang et al. (1999) is modified by incorporating shortage in which c_3 = 2.5 c_2 and c_3 = 7.5 c_2. The detail parameters are provided in Table 3. It is noted that the demand function are in the form $f(t) = a + bt + ct^2$.

Table 3 The 12 sample problems

No.	*a*	*b*	*c*	*H*	c_1	c_2
1	0	900	100	1	9	2
2	0	900	100	2	9	2
3	0	100	5	4	100	2
4	0	1,600	100	3	42	0.56
5	6	1	0.005	11	30	1
6	6	1	0.005	11	50	1
7	6	1	0.005	11	60	1
8	6	1	0.005	11	70	1
9	6	1	0.005	11	90	1
10	100	150	10	1	30	2
11	100	150	10	1.5	30	2
12	100	150	10	2	30	2

Source: Yang et al. (1999)

With the similar steps as illustrated before, the replenishment schedules for 12 sample problems can be determined. The sensitivity analysis with varying values of *n* is also conducted. The complete results for both c_3 = 2.5 c_2 and c_3 = 7.5 c_2 are presented in Table 4.

Table 4 Results of the proposed technique

Problem no.	$c_3 = 22.5\ c_2$			$c_3 = 27.5\ c_2$		
	*n**	*Obj.function*	*Comp.time*)*	*n**	*Obj.function*	*Comp.time*)*
1	6	110.5820	0.2813	7	121.8685	0.8750
2	18	311.4106	7.8281	20	345.7633	20.9531
3	5	1,018.0000	0.2500	6	1,120.2000	0.7188
4	11	874.4733	2.0625	12	969.1520	5.2656
5	4	256.4158	0.1406	5	278.9000	0.4063
6	4	336.4158	0.1563	4	363.3549	0.2500
7	3	368.5417	0.0469	3	402.9584	0.1563
8	3	398.5417	0.0781	3	432.9584	0.1250
9	3	458.5417	0.0781	3	492.9584	0.1406
10	3	136.8213	0.0469	3	146.0815	0.0938
11	4	215.1204	0.1406	4	234.1304	0.2031
12	5	307.6002	0.2031	6	337.2550	0.5000

Unlike the result from the previous example in which the best value of *n* is exactly the same as the predefined value determined from the formula of Teng (1996), in these sample problems it has been found that the best values of *n* is not always the same as the predefined value. A detail analysis for sample problem no. 6 with the predefined value *n* = 3 is presented in Table 5.

Table 5 Sensitivity analysis on the effect of *n* of sample problem 6 at c_3 = 2.5 c_2

I	*n = 3*		*n = 4*		*n = 5*	
	s_i	t_i	s_i	t_i	s_i	t_i
1	0	3.3535	0	2.5358	0	2.0438
2	4.6949	7.4585	3.5501	5.6979	2.8614	4.6280
3	8.5639	1	6.5571	8.4746	5.3346	6.9278
4			9.2416	1	7.5651	9.0354
5					9.6235	1
TC	338.5417		*336.4158*		356.8968	

From the results presented in Table 5, it can be seen that even though the predefined value of *n* is 3, but the sensitivity analysis shows that the lowest inventory cost is achieved when *n* = 4.

6 Discussion

The numerical experiments result on sample problem 1, which is taken from Yang (2006), is showing that the performance of the proposed algorithm is exactly the same as the performance of the exact solution procedure based on backward recursive algorithm proposed by Yang (2006). The result presented in Table 1 is showing that the total cost from the proposed technique is the same as Yang's (2006).

Table 6 Results of the proposed technique and Chen et al.'s (2007) technique at $c_3 = 2.5\ c_2$

Problem no.	*Objective function*				*Computational time *) (second)*	
	*n**	*Proposed technique*	*n**	*Chen et al. (2007)*	*Proposed technique*	*Chen et al. (2007)*
1	6	110.5820[a]	6	110.5820	0.2813[c]	16.1563
2	18	311.4106[b]	15	326.5964	7.8281[c]	244.5938
3	5	1,018.0000[b]	5	1,018.0421	0.2500[c]	3.1719
4	11	874.4733[b]	11	886.7435	2.0625[c]	131.1719
5	4	256.4158[a]	4	256.4158	0.1406[c]	1.4531
6	4	336.4158[a]	4	336.4158	0.1563[c]	1.5156
7	3	368.5417[a]	3	368.5417	0.0469[c]	0.8125
8	3	398.5417[a]	3	398.5417	0.0781[c]	0.8906
9	3	458.5417[a]	3	458.5417	0.0781[c]	0.9219
10	3	136.8213[a]	3	136.8213	0.0469[c]	0.3750
11	4	215.1204[a]	4	215.1204	0.1406[c]	1.4063
12	5	307.6002[a]	5	307.6002	0.2031[c]	2.6406

Table 7 Results of the proposed technique and Chen et al.'s (2007) technique at $c_3 = 7.5\ c_2$

Problem no.	*Objective function*				*Computational time *) (second)*	
	*n**	*Proposed technique*	*n**	*Chen et al. (2007)*	*Proposed technique*	*Chen et al. (2007)*
1	7	121.8685[a]	7	121.8685	0.8750[c]	31.7969
2	20	345.7633[b]	16	368.2256	20.9531[c]	352.4531
3	6	1,120.2000	6	1,120.1995	0.7188[c]	4.1406
4	12	969.1520[b]	11	984.8401	5.2656[c]	98.4844
5	5	278.9000[a]	5	278.9000	0.4063[c]	3.0469
6	4	363.3549[a]	4	363.3549	0.2500[c]	2.3438
7	3	402.9584[a]	3	402.9584	0.1563[c]	0.6406
8	3	432.9584[a]	3	432.9584	0.1250[c]	0.6250
9	3	492.9584[a]	3	492.9584	0.1406[c]	0.5938
10	3	146.0815[a]	3	146.0815	0.0938[c]	0.1201
11	4	234.1304[a]	4	234.1304	0.2031[c]	1.2656
12	6	337.2550[a]	6	337.2550	0.5000[c]	5.0625

The numerical experiments result on sample problem 2 are presented in order to demonstrate the applicability of the proposed algorithm to solve various non-linear demand pattern and the superiority of the proposed algorithm over an existing method for solving the same problem, that is the technique based on Nelder-Mead algorithm developed by Chen et al. (2007). The best total inventory cost for each sample problem and its associated value of *n* and comparison between results obtained from the proposed technique with the results by using Chen et al.'s (2007) technique at different values of shortage cost c_3 are presented in Tables 6 and 7. It is noted that in those tables, the computational times (*) are recorded on a PC with Intel P4 3.4 GHz and 1 GB RAM. Also, specific superscripts are being used for highlighting the results: ([a]) indicates that the proposed technique gives the similar objective function as the Chen et al. (2007) technique, ([b]) indicates that the proposed technique gives the smaller objective function than the Chen et al. (2007) technique, and ([c]) indicates that the proposed technique is faster than the Chen et al. (2007) technique.

Based on the result presented in Table 6 and 7, it can be seen that the proposed technique performs better than Chen et al.'s technique, especially for the problem with large number of replenishment *n*, i.e., $n > 10$. Moreover, the proposed technique requires less computational time than Chen et al.'s.

7 Conclusions

In the research of this paper, a repetitive forward rolling technique has been proposed to solve the inventory decision problem for the case of non-linear increasing demand pattern considering shortage under IFS policy. The proposed technique is applied to help determine the replenishment times and the shortage points in a predefined planning horizon in order to minimise total inventory cost. In addition the proposed repetitive forward rolling technique always ensure that demand over pre-established time horizon is always fulfilled. From numerical experiments, it can be concluded that the proposed technique gives the same result as Yang's (2006). More over it can be applied to more general non-linear increasing demand pattern. Compare to the Nelder-Mead algorithm provided by Chen et al. (2006) the proposed technique give the same or better solution with less computational time. It is also noted that the optimal number of replenishment *n* obtained by Teng (1996) cannot be applied directly to find the optimal number of replenishment for the case of non-linear increasing demand pattern. Based on the numerical experiment that are presented in this paper it is shown that the sensitivity analysis need to be done in order to find the number of replenishment that result the lowest total inventory cost.

Since the research in this paper is only addressing problem with simple shortage case, the future works can be done by applying and modifying the proposed technique for solving more complex inventory policy problem with general non-linear increasing demand pattern, by considering others factors such as partial backlog case, deterioration rate for perishable product, and time value of money.

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