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A Novel Space-time Discontinuous Galerkin Method for Solving of One-dimensional Electromagnetic Wave Propagations

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Abstract

In this paper we propose a high-order space-time discontinuous Galerkin (STDG) method for solving of one-dimensional electromagnetic wave propagations in homogeneous medium. The STDG method uses finite element Discontinuous Galerkin discretizations in spatial and temporal domain simultaneously with high order piecewise Jacobi polynomial as the basis functions. The algebraic equations are solved using Block Gauss-Seidel iteratively in each time step. The STDG method is unconditionally stable, so the CFL number can be chosen arbitrarily. Numerical examples show that the proposed STDG method is of exponentially accuracy in time.

Keywords*: Space-time discontinuous Galerkin, electromagnetic wave*

1. Introduction

The The finite element method has had a greater impact on theory and application on numerical methods during the twentieth century [1]. It has been used throughout the fields of engineering. The success of finite element method is lied on its flexibility. It can be used for solving problem with complex shape and arbitrary boundary conditions.

The achievement of the FE method is the application of the method in computational electromagnetics (CEM) remarkable, especially ini static and quasi static electromagnetics [2, 3] however, the method has not produce entirely satisfactory results in time domain CEM. The governing equations of electromagnetic wave propagation are often discretized with FE method in time domian, resulting in semi algebraic equations in time, which are solved with explicit Runge Kutta family schemes or finite difference method. Explicit methods which do not require large matrix inversion were used for modeling wave propagation [4]. Difficulties of explicit methods are low order, lacking of high-frequency dissipation, and conditionally stable. In order to maintain the calculation stable, a small step size below CFL limit must be kept. Reference [5] and [6] proposed time-discontinuous Galerkin (DG) method for discretizing the temporal domain, while the spatial domain still discretized by conventional FE method. They showed that the results are very promising and the 3^{th} order temporal discretization can be achieved.

Reference [7] developed space-time discontinuous Galerkin (STD) method for solving 1-D scalar wave propagation problems. The spatial and temporal domains are coupled strongly, 1-D wave propagation problems become 2-D problems because variable time is added as second spatial coordinate. They used unstructured triangular space-time element. They showed that $(p+1)^{th}$ order accuracy in L_2 norm can be achieved for arbitrary basis function order (p). While this approach having good accuracies, it has diffulties in implementation of coding and extension to real 3-D problems. Reference [8] continued the work of [5] for solving elastodynamics problems, DG method was used not only for temporal domain but spatial domain also. The elastodynamics equations are transformed into velocity-displacement formulation, so the order of equations are second order in space and first order in time. The results showed that high-order accuracy an be achieved and high–frequency oscillation can be damped.

In this paper, we propose a new space-time discontinuous Galerkin method for solving one-dimensional Maxwell's equations, which can describe electromagnetic wave propagation.

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The spatial and temporal domains are coupled strongly, and the high-order basis functions are tensor product of 1-dimensional Jacobi basis functions in space and time.

2. Governing Equations

In this paper, the 1-D system of transverse electric (TE) Maxwell's equations have been chosen.

$$\varepsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} \quad ; \quad \mu \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} \tag{1}$$

where H_z is magnetic field, E_y is electric field, ε is permittivity μ is permeability of medium. In this paper ε and μ are kept constant and set to be equal one.

For simplicity, the governing equations are written in vector notation:

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{q}}{\partial x} = 0 \tag{2}$$

where $\mathbf{q} = \begin{bmatrix} H_z & E_y \end{bmatrix}^T$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3. Numerical Method

3.1. Discretization

Inspired by work of Maerschalck [9], we extended the work to solve 1-dimensional STDG formulation. In the space-time formulation of equation (2), the variable *t* is considered as a third spatial variable. The dimension of the discretization is always one higher than dimension of the actual governing equations.

In our approach we divided the space-time domain into rectangular slabs $D^k = \Omega \times \Delta t$, where Ω is the spatial domain and $\Delta t = (t_n, t_{n+1})$ are the time step, see Figure 1.

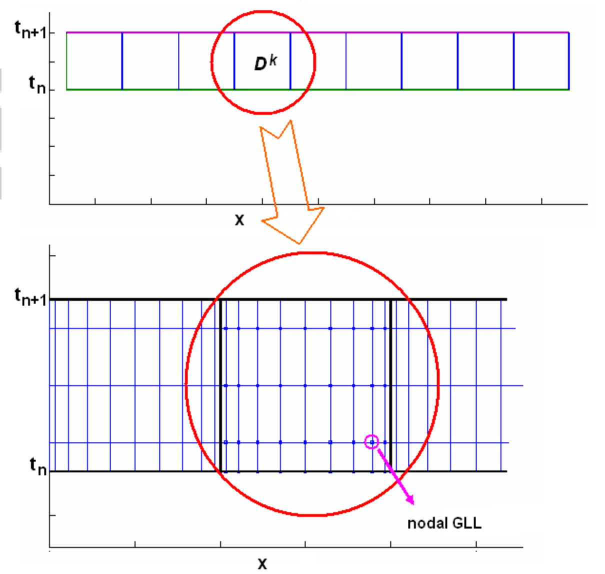

Figure 1. One-dimensional space-time slabs

The physical space-time elements are mapped into reference elements as shown in Figure 2 using mapping function:

$$\Psi(\mathbf{r})=\mathbf{x}=\begin{pmatrix}x\\t\end{pmatrix}=\frac{(1-r)(1-u)}{4}\begin{pmatrix}x^1\\t^1\end{pmatrix}+\frac{(1+r)(1-u)}{4}\begin{pmatrix}x^2\\t^2\end{pmatrix}+\frac{(1-r)(1+u)}{4}\begin{pmatrix}x^3\\t^3\end{pmatrix}+\frac{(1+r)(1+u)}{4}\begin{pmatrix}x^4\\t^4\end{pmatrix}$$

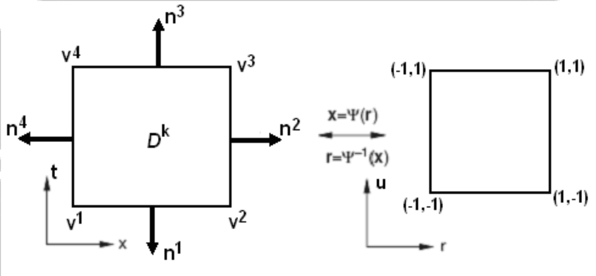

Figure 2. Mapping between physical element and standard element

For discretization, the governing equations are integrated by parts using Galerkin weighted residual method in each space-time element to obtain a discrete form of the problem.

$$\left(l_n,\frac{\partial l_m}{\partial t}\right)_{D^k}\mathbf{q}_m+\mathbf{A}\left(l_n,\frac{\partial l_m}{\partial x}\right)_{D^k}\mathbf{q}_m-(l_n,l_m)_{D^k}\mathbf{f}_m -(\mathbf{I}n_t+\mathbf{A}n_x)(l_n,l_m)_{\partial D^k}\left(\mathbf{q}_m-\mathbf{q}^*\right)=0 \quad (3)$$

where l_m is nodal space-time basis function which is tensor product of Jacobi polinomial, n_t is temporal normal vector, n_x are the spatial normal vector and $\mathbf{q}^*$ is numerical flux. In this paper, Lax-Friedrich flux was used for spatial flux and upwind flux for temporal flux. $\mathbf{q}$ is local solution which is approximated by:

$$\mathbf{q}_h^k(\mathbf{x})=\sum_{i=1}^{N_p}\mathbf{q}_h^k\left(\mathbf{x}_i^k\right)l_i^k(\mathbf{x}) \quad (5)$$

The discrete form of equation (3) in a space-time element can be written as follows:

$$\begin{pmatrix}\mathbf{A}_t & \mathbf{A}_x\\ \mathbf{A}_x & \mathbf{A}_t\end{pmatrix}\begin{pmatrix}E_y\\H_z\end{pmatrix}=\begin{pmatrix}\mathbf{R}_a\\ \mathbf{R}_b\end{pmatrix} \quad (4)$$

$$\mathbf{A}_t=\mathbf{M}\frac{\partial u}{\partial t}\mathbf{D}_u+\mathbf{L}_b+\frac{1}{2}\mathbf{L}_s$$

$$\mathbf{A}_x=-\mathbf{M}\frac{\partial r}{\partial x}\mathbf{D}_r+\frac{1}{2}\mathbf{L}_s$$

$$M_{ij}=\int_{D^k}l_i(\mathbf{x})l_j(\mathbf{x})d\mathbf{x}=J^k\int_I l_i(\mathbf{r})l_j(\mathbf{r})d\mathbf{r}$$

$$(D_r)_{ij} = \frac{\partial l_j(r_i)}{\partial r} \; ; \; (D_u)_{ij} = \frac{\partial l_j(r_i)}{\partial u}$$

where $\mathbf{L}_b$ is the matrix containing temporal flux, $\mathbf{L}_s$ is the matrix containing Lax-Friedrich flux, and $(\mathbf{R}_a, \mathbf{R}_b)$ are residual matrices. The discrete form of Equation (4) is solved iteratively, element by element, by using Block Gauss Seidel method in each time step.

4. Results and Discussion

In this section, we demonstrate the performance of the STDG method. The following initial conditions are taken to perform numerical simulation:

$$\begin{aligned} &H_z(x,0) = -\cos(2\pi x)\cos(2\pi y) \\ &E_y(x,0) = 0 \\ &0 \le x \le 1 \end{aligned} \tag{5}$$

The exact solution for the electromagnetic fields are:

$$\begin{aligned} &H_z(x,t) = -\cos(2\pi x)\cos(2\pi t) \\ &E_y(x,t) = -\sin(2\pi x)\sin(2\pi t) \end{aligned} \tag{6}$$

We will discussed the rate of convergence and accuracy of the STDG method. We tried to examine the order of STDG formulation in time direction only, examination of the order in space direction can be found in [10]. The spatial domain is divided into 20 elements, the spatial polynomial order is kept constant N_s = 4 and temporal polynomial order is varied from N_t = 0 to N_t = 3. We used the L_2 norm as the error indicator which is defined as:

$$\|\varepsilon\|_{L^2} = \left(\int_0^L \left(E_{y_{exact}} - E_{y_{STDG}}\right)^2 dx + \int_0^L \left(H_{z\,exact} - H_{z\,STDG}\right)^2 dx \right)^{\frac{1}{2}}$$

We compared the error solution for varying of temporal order (N_t) on final time $t = T$. For all N_t the solution is calculated for a fixed time level $T = 2.4$, except for N_t = 3 the final time T is equal 2.1. The Courant-Friedrich-Lewy (CFL) number is defined as:

$$CFl = \frac{\Delta t}{\Delta x_{\min}}$$

where $\Delta x_{\min}$ is the distance between two closest points. The values of Δt for different N_t are listed in Table 1.

Table 1. The values of Δt, Δx and CFL

N_t =0		N_t =1		N_t =2		N_t =3	
Δt	*CFL*	Δt	*CFL*	Δt	*CFL*	Δt	*CFL*
0.004	0.46	0.05000	5.79	0.200	23.17	0.7000	81.08
0.0020	0.23	0.02500	2.90	0.100	11.58	0.3500	40.54
0.0010	0.12	0.01250	1.45	0.050	5.79	0.1750	20.27
0.0005	0.06	0.00625	0.72	0.025	2.90	0.0875	10.13

The snapshots of electromagnetic fields are shown by Figures 3(a-b). Those figures described the evolution of the electromagnetic fields, actually they are standing waves. They did not propagate but oscillated.

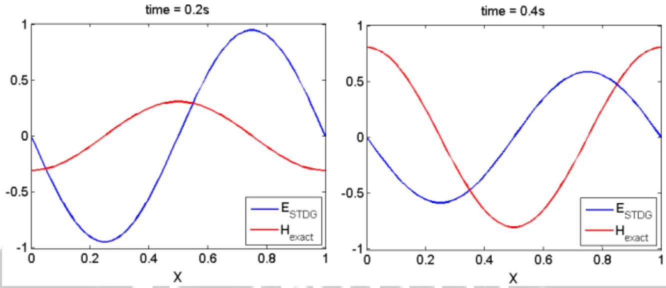

Figure 3(a). Mesh for second example

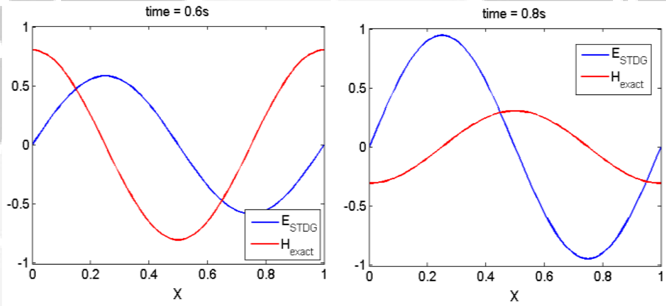

Figure 3(b). Mesh for second example

Figures 4(a) and 4(b) shows the history of convergence for various temporal polynomial order. From those figures, it can be inferred that the error can reduced sigificantly by increasing the order of temporal polynomial. The error increases almost linearly, it indicates that error grows slowly and numerical schemes are stable for long time calculations

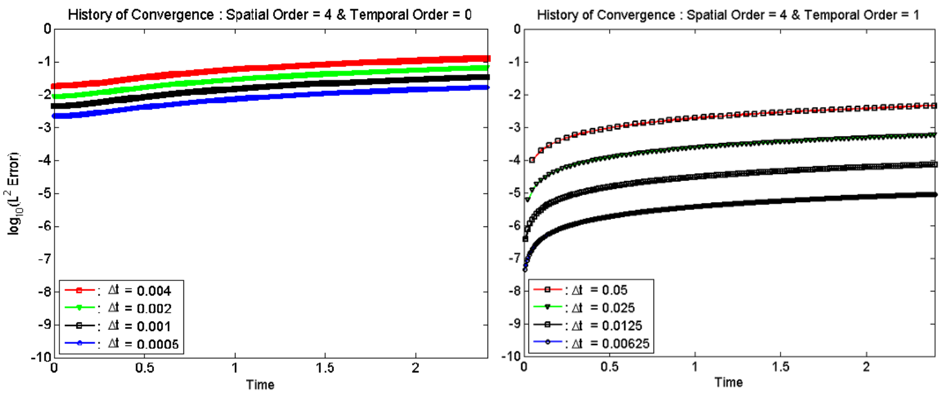

Figure 4(a). History of convergence for N_t =0 and N_t =1

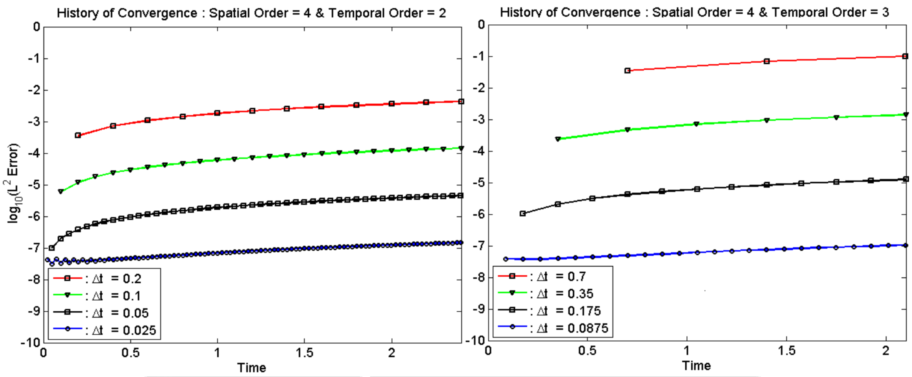

Figure 4(b). History of convergence for N_t =2 and N_t =3

Figure 5 indicates that optimal convergence rates of $O\left(\Delta t^{2N_t+1}\right)$ are achieved.

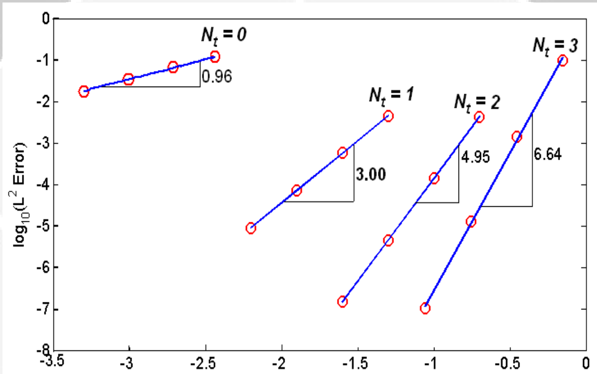

Figure 5. L_2 norm vs. Δt

5. Conclusion

We have developed a new space-time discontinuous galerkin (STDG) method for simulation of the one-dimensional electromagnetic wave fields. The new method has high accuracy, exponential orde accuracy of $O\left(\Delta t^{2N_t+1}\right)$can be achieved. Also, the new method can adopt a larger time step without any instabilities.

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