VOLUME IX, NOMOR 2, APRIL 2005 ISSN 1410-5004

JURNAL TEKNOLOGI INDUSTRI

JTI	Vol. IX	No. 2	Hal. 85-182	Yogyakarta April 2005	ISSN 1410-5004

UNIVERSITAS ATMA JAYA YOGYAKARTA
Fakultas Teknologi Industri

JURNAL TEKNOLOGI INDUSTRI
ISSN 1410-5004

AKREDITASI
Keputusan
Direktur Jendral Pendidikan Tinggi
Departemen Pendidikan Nasional
Nomor: 23a/DIKTI/Kep/2004

Layanan *online* internet tersedia dengan alamat: http://fti.uajy.ac.id/jurnal

Alamat Redaksi
Tata Usaha Fakultas Teknologi Industri
Universitas Atma Jaya Yogyakarta
Jln. Babarsari No. 43, Yogyakarta 55281
Telp. (0274) 487711 Fax. (0274) 485223
E-mail: jti@mail.uajy.ac.id
Home page: http://fti.uajy.ac.id/jurnal

Jurnal Teknologi Industri diterbitkan oleh Fakultas Teknologi Industri Universitas Atma Jaya Yogyakarta sebagai media untuk menyalurkan pemahaman tentang aspek-aspek teknologi baik teknologi industri maupun teknologi informasi berupa hasil penelitian lapangan atau laboratorium maupun studi pustaka. Jurnal ini terbit empat kali dalam setahun yaitu pada bulan **Januari**, **April**, **Juli**, dan **Oktober**. Redaksi menerima sumbangan naskah dari dosen, peneliti, mahasiswa maupun praktisi dengan ketentuan penulisan seperti tercantum pada halaman dalam sampul belakang.

Distribusi
Pusat Pemasaran Universitas (PPU)
Universitas Atma Jaya Yogyakarta
Gedung Don Bosko
Jln. Babarsari No. 5, Yogyakarta 55281
Telp. (0274) 487711 Fax. (0274) 487748
E-mail: ppu@mail.uajy.ac.id

Biaya Berlangganan

Langganan	Rp 200.000,00/tahun
Eceran	Rp 60.000,00/nomor

Biaya Penulisan
Bagi penulis yang naskahnya diterbitkan, penulis diwajibkan membayar biaya sebesar Rp 500.000,00 per naskah (sudah termasuk biaya berlangganan selama 1 tahun).

Rekening (*Bank Account*)
Bank BNI 46 a.n. UAJY:
228.007.121.001.001

ISSN 1410-5004

JURNAL TEKNOLOGI INDUSTRI

Volume IX Nomor 2 April 2005

DAFTAR ISI

High Order Discontinuous Galerkin for Numerical Simulation of Elastic Wave Propagation

Pranowo, F. Soesianto, dan Bambang Suhendro
Program Studi Teknik Informatika, Fakultas Teknologi Industri, Universitas Atma Jaya Yogyakarta
Jln. Babarsari No. 43 Yogyakarta 55281
E-mail: pran@mail.uajy.ac.id

Abstract

We present a study of elastic wave propagation in isotropic media. The Discontinuous Galerkin Method is applied to solve the elastodynamic equations which represent elastic wave propagation. The elastodynamic equations are transformed into a stress-velocity formulation. The Discontinuous Galerkin Method is a finite element that allows a discontinuity of the numerical solution at element interface. Through a proper choice of the flux computation points, the method only requires communication between elements that have common faces. The utilization of high-order Legendre polynomials as basis functions has been shown to be more efficient in reducing the numerical dispersion and numerical dissipation. Discontinuous Galerkin Method is a compact method, high-order basis functions can be used easily without any essentially difficulty and even spectral accuracy becomes obtainable. Temporal discretization utilized explicit staggered leapfrog method. We compare the numerical results to the exact solutions and the comparison shows a good agreement.

Keywords: elastic wave propagation, discontinuous Galerkin, high order basis functions

1. Introduction

The discontinuous galerkin (DG) method originally developed by Reed and Hill for the solution of neutron transport problem. Le Saint and Raviart were the first to put the method on a firm mathematical base Cockburn and Shu (1997; 1998). The technique lay dormant for several years before becoming popular. Cockburn and Shu (1997; 1998) developed DG method to solve convection-diffusion problem and extended it to multidimensional systems case. Recently, the DG formulation of the finite element method has been increasingly use in computational fluid dynamics (CFD) (Karniadakis & Sherwin, 1999; Van Der Vegt & Van der Ven, 1998), computational electromagnetic (CEM) (Hesthaven & Warburton, 2001) and computational aeroacoustics (CAA) (Atkins & shu, 1996; Atkins, 1997).

It is well known that highly accurate methods are required for long time simulations of wave propagation, which are essentially non dispersive and non dissipation (Hu & Rasetarinera, 1999). Many numerical schemes such as finite difference and finite element are developed to study wave propagation. Among the schemes, the DG method provides an attractive approach to solve problems containing discontinuities, such as those arise in hyperbolic systems. The DG method allows more general mesh configuration and interelement continuity is not required. The basis function is discontinuous across mesh boundaries. Through a proper choices of flux computation points, the method only requires communication between mesh that have common faces. No global matrix inversion is required and the problem can be solved locally in each mesh. The DG method can be regarded as cross between a finite volume and finite element method and it has many of the good properties of both.

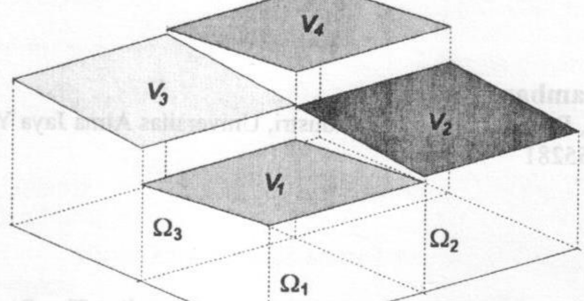

Figure 1. Mesh and Solution of Discontinuous Galerkin Method

Lowrie (1996) considered the space-time DG method which involves discontinuous mesh in both space and time to solve hyperbolic laws (Lowrie, 1996). The method requires excessive resources to be useful for practical application. Atkins and Shu (1996) described a quadrature-free formulaton for DG method. They apply the formulation to linearized Euler equations. Stanescu *et al.* combined DG method with spectral element method to solve aircraft engine noise scattering (Stanescu *et al.*). They used high-order Legendre polynomials as basis functions. The numerical results show that trends of the noise field are well predicted. Li (1996) developed adaptive space-time DG method to solve elastodynamic equations. Triangular mesh and linear basis function were used. The numerical results have a good agreement with exact solution.

In this paper, elastodynamic equations, which described elastic wave propagation, are solved using DG method. High order Legendre polynomials are used as basis functions. The elastodynamic equations will be discretized using rectangular mesh and explicit Leapfrog method is used as time integration method.

2. Governing Equations

The Elastodynamic Equations are set of linear hyperbolic equations. The equations are transformed into the following first-order hyperbolic system via stress-velocity formulation.

$$\frac{\partial v_x}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + f_x \tag{1.a}$$

$$\frac{\partial v_y}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) + f_y \tag{1.b}$$

$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \mu\frac{\partial v_y}{\partial y} \tag{1.c}$$

$$\frac{\partial \tau_{yy}}{\partial tt} = (\lambda + 2\mu)\frac{\partial v_y}{\partial y} + \mu \frac{\partial v_x}{\partial x} \tag{1.d}$$

$$\frac{\partial \tau_{xz}}{\partial t} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{1.e}$$

In which (v_x, v_y) is the velocity vector, $(\tau_{xx}, \tau_{yy}, \tau_{xy})$ is the stress tensor, (f_x, f_y) is the body force vector, ρ is the density , λ and μ are Lame coefficient. Stress-velocity formulation for one-dimensional propagation in the x direction are:

$$\frac{\partial v_y}{\partial t} - \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial x} = 0 \tag{2.a}$$

$$\frac{\partial \tau_{xy}}{\partial t} - \mu\frac{\partial v_y}{\partial x} = 0 \tag{2.b}$$

3. DG Discretization

a. One Dimensional Discretization

We present one-dimensional DG Discretization in one dimension to begin with, and extend them to two dimensions. We divide the interval of domain into subintervals $\Omega_j = \lfloor x_j, x_{j+1} \rfloor$, $j = 0,...N$. In each subinterval (element), we expand the velocity and stress in terms of Legendre cardinal functions:

$$\begin{aligned} v_y &= \sum_{i=0}^{m} h_i(x)(v_y)_i = [h][v_y] \\ \tau_{xy} &= \sum_{i=0}^{m} h_i(x)(\tau_{xy})_i = [h][\tau_{xy}] \end{aligned} \tag{3}$$

The Legendre cardinal functions are written as below:

$$h_i(x) = \frac{(x^2 - 1)L_m'(x)}{m(m+1)L_m(x_i)(x - x_i)} \tag{4}$$

L_m are m-th order Legendre Polynomials.

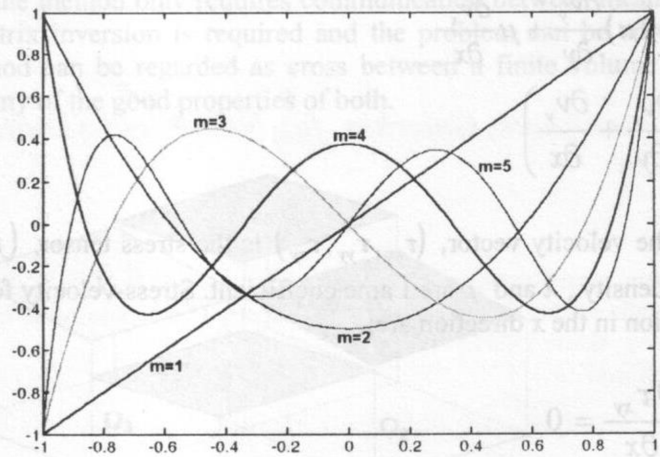

Figure 2. Legendre Polynomials

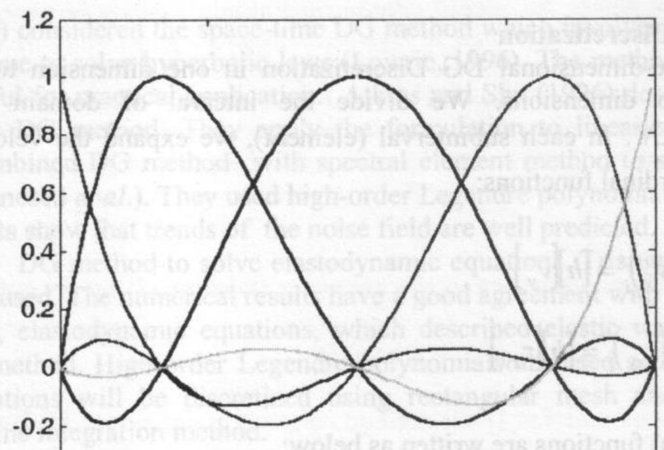

Figure 3. Fourth Order Legendre Cardinal Function

As example, we sample equation (2.a) according Galerkin's procedure using $[h]$ as testing function:

$$\int_{x_j}^{x_{j+1}} [h]^T [h] \frac{\partial [v_y]}{\partial t} dx - \frac{1}{\rho} \int_{x_j}^{x_{j+1}} [h]^T \frac{\partial [h]}{\partial x} [\tau_{xy}] dx = 0 \tag{5}$$

Integrate by parts the term with spatial derivative:

$$\int_{x_j}^{x_{j+1}} [h]^{\mathrm{T}} [h] \frac{\partial [v_y]}{\partial t} dx - \frac{1}{\rho} \begin{pmatrix} -\int_{x_j}^{x_{j+1}} \frac{\partial [h]^{\mathrm{T}}}{\partial x} [h][\tau_{xy}] dx \\ + [h]^{\mathrm{T}} [h][\tau_{xy}]_{x_j}^{x_{j+1}} \end{pmatrix} = 0 \tag{6}$$

We make a modification to approximate $[\tau_{xy}]_{x_j}^{x_{j+1}}$ by $[\hat{\tau}_{xy}]_{x_j}^{x_{j+1}}$, integrate by parts the second term:

$$\int_{x_j}^{x_{j+1}} [h]^{\mathrm{T}} [h] \frac{\partial [v_y]}{\partial t} dx - \frac{1}{\rho} \begin{pmatrix} \int_{x_j}^{x_{j+1}} \frac{\partial [h]^{\mathrm{T}}}{\partial x} [h][\tau_{xy}] dx \\ + [h]^{\mathrm{T}} [h][\hat{\tau}_{xy} - \tau_{xy}^{-}]_{x_j}^{x_{j+1}} \end{pmatrix} = 0 \tag{7}$$

The numerical flux $(\hat{\tau}_{xy})$ at the interface between elements is approximated by using average flux:

$$\hat{\tau}_{xy} = \frac{1}{2}\left(\tau_{xy}^{+} + \tau_{xy}^{-}\right) \tag{8}$$

The (+) notation implies the limit of τ_{xy} from outside of the element and the (-) notation implies the limit of τ_{xy} from inside of the element.

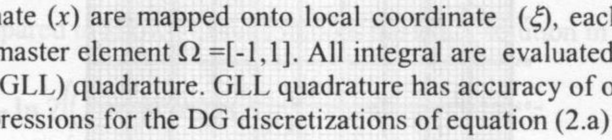

Figure 4. Flux and Local Coordinate

The global coordinate (x) are mapped onto local coordinate (ξ), each element $\Omega = [x_j, x_{j+1}]$ is then mapped onto master element $\Omega = [-1,1]$. All integral are evaluated numerically using Gauss Lobatto Legendre (GLL) quadrature. GLL quadrature has accuracy of order ($2m$-2).

The final expressions for the DG discretizations of equation (2.a) is:

$$\mathbf{M}\frac{\partial [v_y]}{\partial t} - \frac{1}{\rho}\mathbf{C}[\tau_{xy}] - \frac{1}{\rho}\left([h]^{\mathrm{T}} [h][\hat{\tau}_{xy} - \tau_{xy}^{-}]_{x_j}^{x_{j+1}} \right) = 0 \tag{9}$$

where : M is mass matrix and C is advection matrix.

$$\mathbf{M} = J\mathbf{W} \; ; \; J = \frac{dx}{d\xi} ; \; \mathbf{C} = \mathbf{WD} \tag{10.a}$$

$$\mathbf{D}=\frac{\partial h_j(\xi_i)}{\partial \xi}\begin{cases}\dfrac{L_m(\xi_i)}{L_m(\xi_j)(\xi_i-\xi_j)} & \text{if } i\neq j\\ 0 & \text{if } i=j, i\neq 0,m\\ \dfrac{-m(m+1)}{4} & \text{if } i=j=0\\ \dfrac{m(m+1)}{4} & \text{if } i=j=m\end{cases} \tag{10.b}$$

$$\mathbf{W}=\begin{bmatrix}\omega_1 & 0 & - & 0\\ 0 & \omega_2 & - & 0\\ - & - & - & -\\ 0 & 0 & - & \omega_m\end{bmatrix} \tag{10.c}$$

J is the Jacobian W is the weight matrix and ω_j are the weights and D is the differential matrix.

b. Two Dimensional Discretization

The domain is divided into non-overlapping rectangular elements within which a high order polynomial expansion is used. We mapped the global coordinates (x,y) onto local coordinates (ξ,η). The basis (trial or test function) is constructed by taking a product of the one-dimensional basis which can be thought of as one-dimensional tensors. We take expansion of v_y as example:

$$v_y=\sum_{k=1}^{m}\sum_{l=1}^{m} v_{ykl}h_k(\xi)h_l(\eta) \tag{11}$$

Figure 5 shows the two-dimensional mesh and figure 6 shows the basis functions. The DG discretization of 2-dimensional case is analogous to the one-dimensional case, for more detail see Karniadakis and Sherwin (1999).

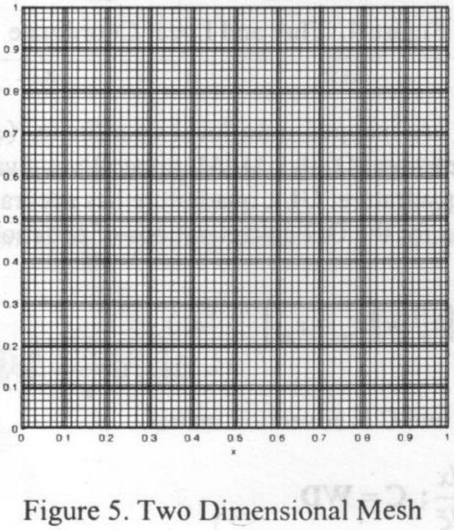

Figure 5. Two Dimensional Mesh

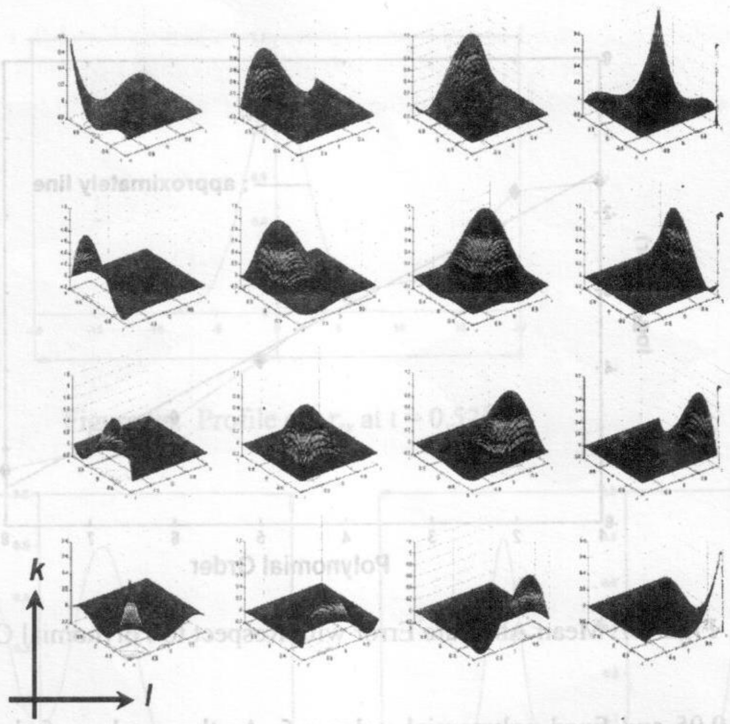

Figure 6. Two Dimensional 3rd Order Basis Functions

4. Numerical Results and Discussion
a. One dimensional Case

To evaluate the accuracy of DG scheme, the following initial conditions are taken to perform numerical simulations:

$$\begin{aligned} &v_y(x,0) = 0 \\ &\tau_{xy}\left(x, \frac{\Delta t}{2}\right) = \exp\left(-\ln 2x^2/9\right) \end{aligned} \quad ; -100 \le x \le 100 \qquad (12)$$

The results are compared to known exact solution, the exact solution in this case is:

$$\tau_{xy}(x,t) = \begin{pmatrix} \exp\left(-\ln 2\left((x-t)^2\right)/9\right) + \\ \exp\left(-\ln 2\left((x+t)^2\right)/9\right) \end{pmatrix} / 2 \qquad (13)$$

We take $\Delta t = 0.05$ and fixed number of elements = 50. Figure 5 shows the mean absolute error (MAE) versus polynomial order. The MAE is plotted on logarithmic scale and we can deduce exponential (spectral) convergence from approximately straight line on the plot.

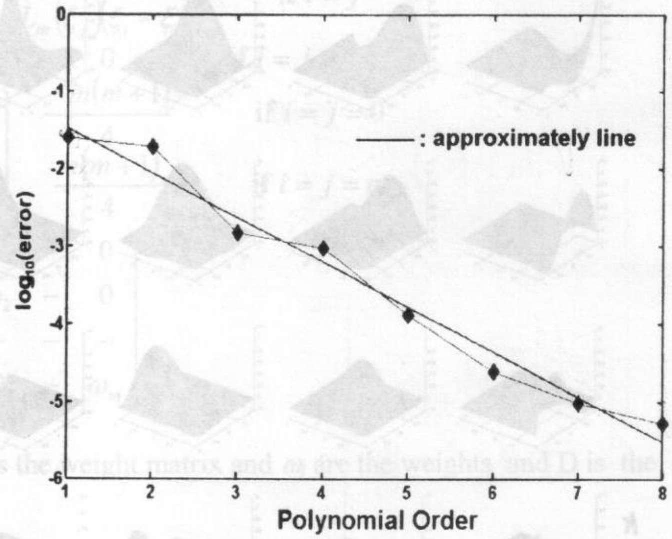

Figure 7. Mean Absolute Error with Respect to Polynomial Order

We take Δt = 0.05 and fixed polynomial order = 5. As the number of elements increases, order of accuracy also increases linearly. Figure 9a-9c show the propagation of the shear wav

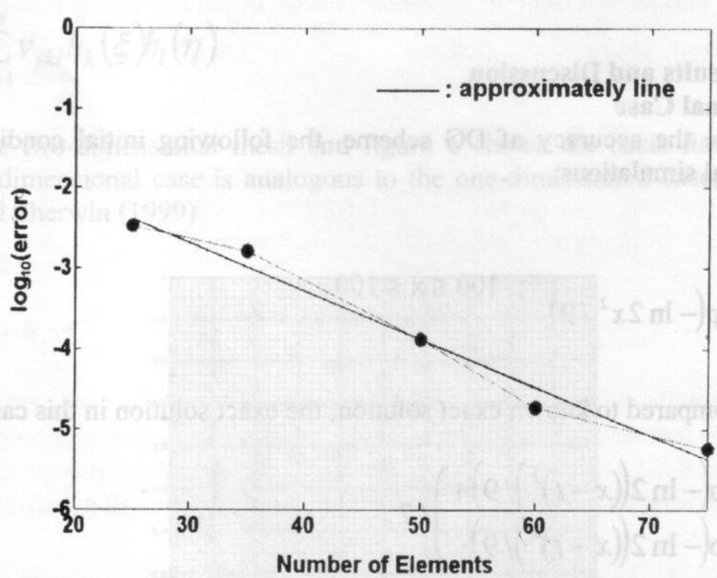

Figure 8. Mean Absolute Error with Respect to Number of Elements

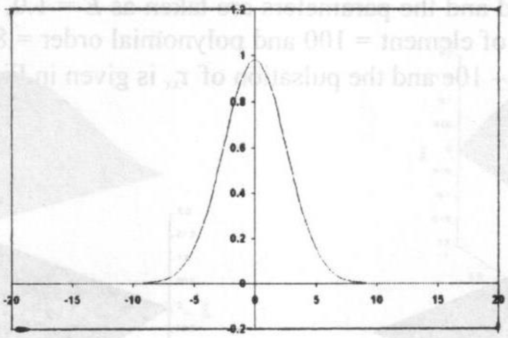

Figure 9a. Profile of τ_{xy} at t = 0.525

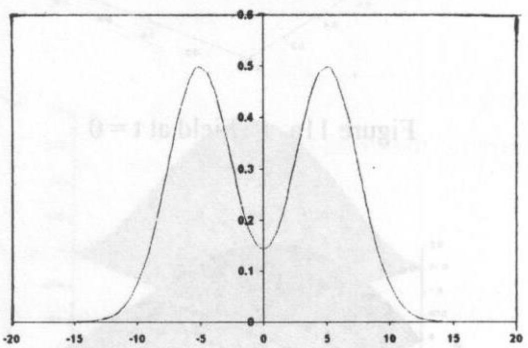

Figure 9b. Profile of τ_x at t = 5.525

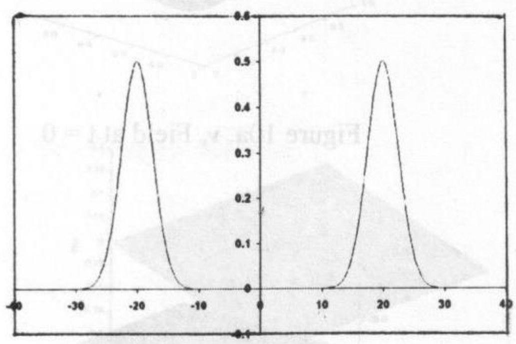

Figure 9c. Profile of τ_x at t = 20.025

b. Two Dimensional Case

We consider a problem defined on the unit square [0,1]×[0,1] with (v_x, v_y)=0 on the boundary. We choose the body force as (Li, 1996):

$$\begin{aligned} f_x &= (\lambda+\mu)(1-2x)(1-2y)Sin\omega t \\ f_y &= \big(\rho\omega^2 xy(1-x)(1-y) - 2\mu y(1-y) \\ &\quad - 2(\lambda+2\mu)x(1-x)\big)Sin\omega t \end{aligned} \tag{14}$$

The exact solutions are:

$$\begin{aligned} v_x &= 0 \\ v_y &= -xy(1-x)(1-y)Cos\,\omega t \\ \tau_{xx} &= -\lambda x(1-x)(1-2y)Sin\,\omega t \\ \tau_{yy} &= -(\lambda+2\mu)x(1-x)(1-2y)Sin\,\omega t \\ \tau_{xy} &= -\mu y(1-y)(1-2x)Sin\,\omega t \end{aligned} \tag{15}$$

Plain strain condition is assumed and the parameters are taken as $E = 1.0$, $\nu = 0.3$, $\rho = 1.0$, ω 1.0, time step = 0.002, numbers of element = 100 and polynomial order = 8. The pulsation of with time is given in Figure 10a – 10e and the pulsation of τ_{xx} is given in Figure 11a – 11e.

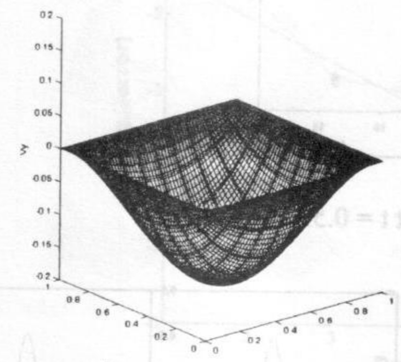

Figure 10a. v_y Field at t = 0

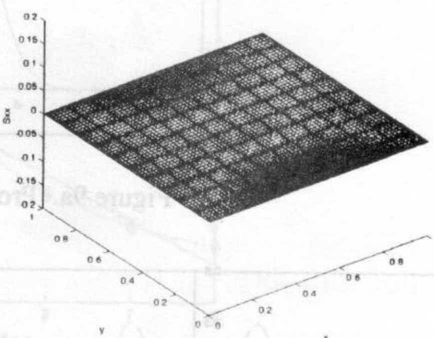

Figure 11a. τ_{xx} Field at t = 0

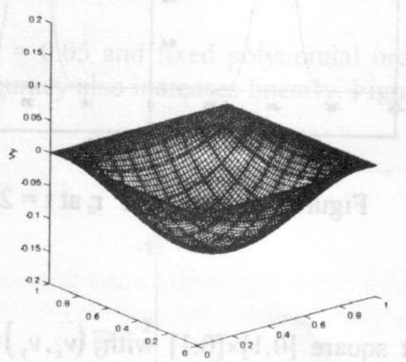

Figure 10b. v_y Field at t = 0.25

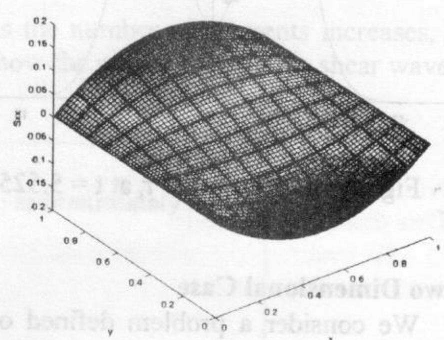

Figure 11b. τ_{xx} Field at t = 0.25

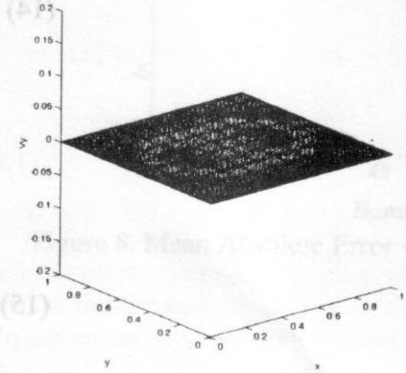

Figure 10c. v_y Field at t = 0.5

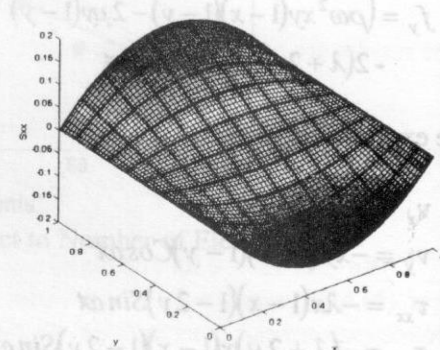

Figure 11c. τ_{xx} Field at t = 0.5

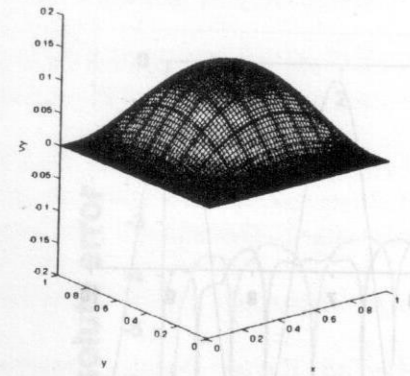

Figure 10d. v_y Field at t = 0.75

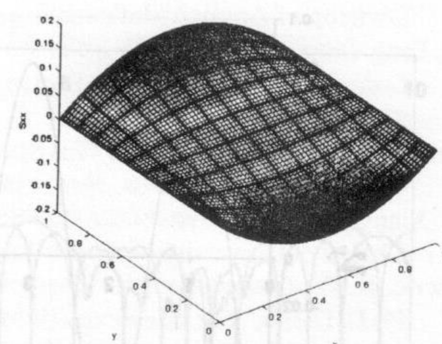

Figure 11d. τ_{xx} Field at t = 0.75

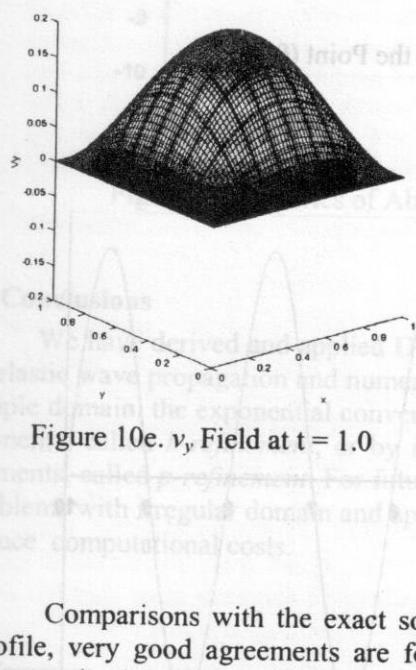

Figure 10e. v_y Field at t = 1.0

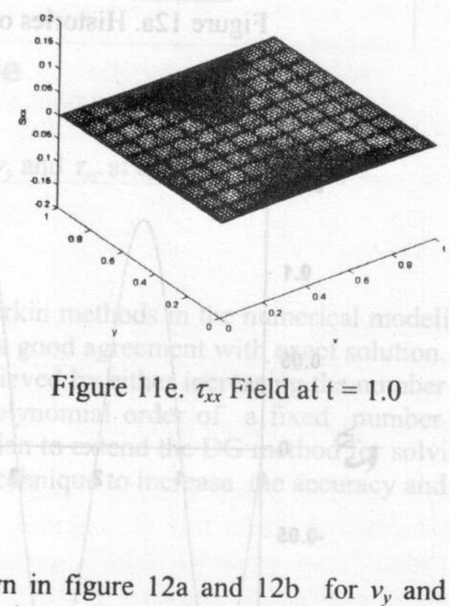

Figure 11e. τ_{xx} Field at t = 1.0

Comparisons with the exact solutions are shown in figure 12a and 12b for v_y and τ_{yy} profile, very good agreements are found. Figure 13 shows that trends of DG errors almost constant in time, the growth of error (dispersive & dissipation error) in DG method can be reduced by using high-order basis.

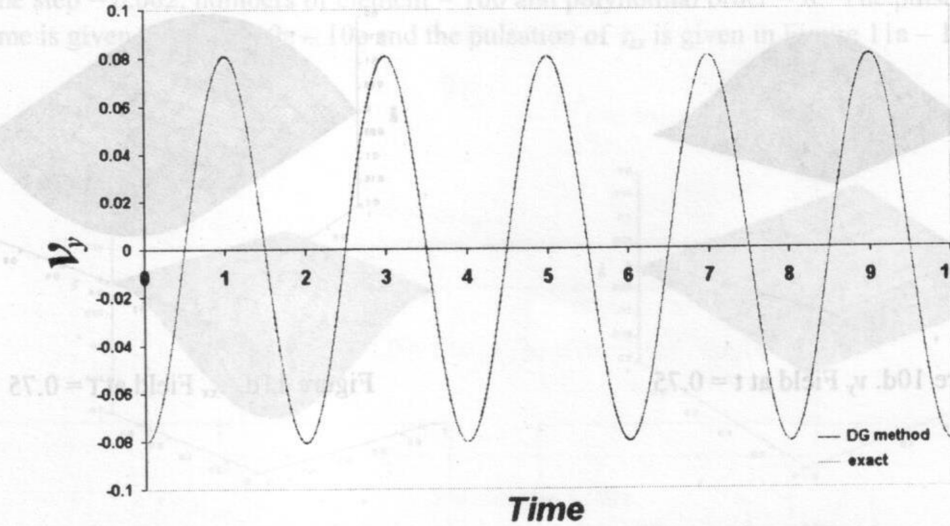

Figure 12a. Histories of v_y Responses at the Point (0.4,0.4)

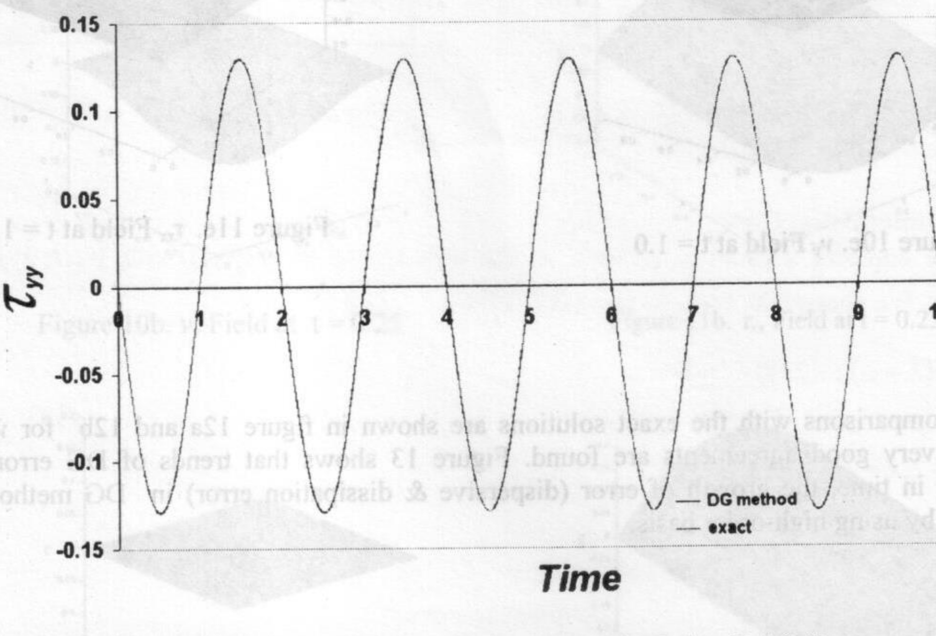

Figure 12b. Histories of τ_{yy} Responses at the Point (0.4,0.4)

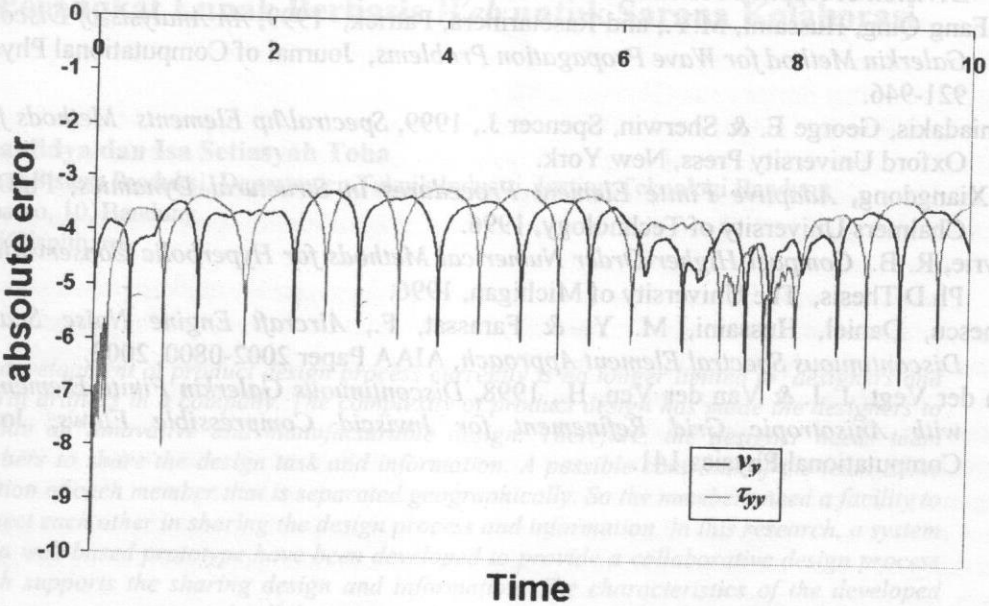

Figure 13. Histories of Absolute Errors of v_y and τ_{yy} at the Point (0.4,0.4)

5. Conclusions

We have derived and applied Discontinuous Galerkin methods in the numerical modeling of elastic wave propagation and numerical results have a good agreement with exact solution. In simple domain, the exponential convergence can be achieved by either increasing the number of elements, called *h-refinement*, or by increasing the polynomial order of a fixed number of elements, called *p-refinement*. For future research, we plan to extend the DG method for solving problems with irregular domain and apply *hp adaptive* technique to increase the accuracy and to reduce computational costs.

References

Atkins, Harold L. and Shu, Chi Wang, 1996, *Quadrature-Free Implementation of Discontinuous Galerkin for Hyperbolic Equations,* NASA AIAA paper 96-1683.

Atkins, Harold L., 1997, *Continued Development of The Discontinuous Galerkin Method for Computational Aeroacoustic Applications* NASA AIAA paper 97-1581.

Cockburn, Bernardo & Shu, Chi Wang, 1997, *The Local Discontinuous Galerkin Method for Time-Dependent Convection-Diffusion System*, ICASE Report 97-32.

Cockburn, Bernardo & Shu, Chi Wang, 1998, *The Runge-Kutta Discontinuous Galerkin Finite Element Method for Conservative. V. Multidimensional Systems*, Journal of Computational Physics 141.

Hesthaven, J. S., Warburton, T., 1998, *High-Order/Spectral Methods on Structured Grids: Time-Domain Solution of Maxwell's Equations,* Research Report BrownSC-2001- Division of Applied Mathematics, Brown University.

Hu, Fang Qing, Hussaini, M.Y., and Rasetarinera, Patrick, 1999, *An Analysis of Discontinuo Galerkin Method for Wave Propagation Problems*, Journal of Computational Physics 15 921-946.

Karniadakis, George E. & Sherwin, Spencer J., 1999, *Spectral/hp Elements Methods for CF* Oxford University Press, New York.

Li, Xiangdong, *Adaptive Finite Element Procedures in Structural Dynamics*, Ph.D Thes Chalmers University of Technology, 1996.

Lowrie, R. B., *Compact Higher-Order Numerical Methods for Hyperbolic Conservation Law* Ph.D Thesis, The University of Michigan, 1996.

Stanescu, Daniel, Hussaini, M. Y. & Farassat, F., *Aircraft Engine Noise Scattering Discontinuous Spectral Element Approach*, AIAA Paper 2002-0800, 2002.

Van der Vegt, J. J. & Van der Ven, H., 1998, *Discontinuous Galerkin Finite Element Meth with Anisotropic Grid Refinement for Inviscid Compressible Flows*, Journal Computational Physics 141.

JURNAL TEKNOLOGI INDUSTRI

Volume IX Nomor 2 April 2005

Volume IX tahun 2005 menerbitkan 4 nomor dalam bentuk cetakan dan publikasi secara *online* di internet. Jurnal Teknologi Industri *Online* dapat diakses lewat internet dengan alamat **http://fti.uajy.ac.id/jurnal**. Fasilitas layanan yang tersedia antara lain: informasi langganan, pengiriman naskah dan layanan melalui e-mail.

Nomor 2

ISSN 1410 - 5004

Akreditasi SK DIRJEN DIKTI
Nomor: 23a/DIKTI/Kep/2004